Applications of Hurst Coefficient Analysis to Chaotic Response of ODE Systems: Part 1a, The Original Lorenz System of 1963

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Abstract

I have coded the process for calculating Hurst coefficients as described on Page 19 of this preprint by D. Koutsoyiannis:

Koutsoyiannis, D., Nonstationarity versus Scaling in Hydrology, *Journal of Hydrology*, **324**, 239–254, 2006. (http://www.itia.ntua.gr/en/docinfo/673/)

The ultimate objectives include applying the analysis method to; (1) results from classic systems of non-linear ordinary differential equations (ODEs) that exhibit chaotic response, (2) measured temperature data, and (3) numbers calculated by a GCM. These notes focus on item (1).

The objectives of looking at the characteristics of the methodology when applied to chaotic response include determining the base-line for the investigations of (2) and (3). The Hurst-coefficient approach might provide a bases for testing GCM calculated numbers relative to comparisons with measured temperature data. That is, within the Hurst-coefficient methodology, can the GCM numbers be shown to have the same Hurst-coefficient properties as that determined by measured data.

The original Lorenz ODE system from 1963, (Reference 1), which is known to exhibit chaotic response is analyzed in this report. The parameter values are taken to be Ra=28.0. Pr=10.0, and b=3.0/8.0. Values of the step size used for numerical integration were 0.001 and 0.01; a systematic study of step size, however, was not conducted.

Application of the Hurst-Coefficient analysis to numerical results from ODEs should provide insight into what the Hurst coefficients should be for purely chaotic systems having constant values for the parameters.

The results of the analyses show; (1) for smaller segment lengths the Hurst coefficient is H = 1.0, (2) for longer segment lengths the Hurst coefficient does not show a constant value, (3) some analyses of longer segment lengths indicate chaotic response to be random numbers. This latter finding is not in agreement with the basis of chaotic response.

NOTE: This report got kind of long. So I have decided to stop at this point and make this report. I have already results for the Lorenz original system using a larger value of the *Ra* parameter. Those results will be the subject of the next notes. Additionally, I have preliminary results for the Lorenz system of 1990, (Reference 2), which includes source terms in the ODEs. I have already made zeroth-order applications to and analysis of (1) measured temperature data, and (2) the results from GCM codes. These are not yet ready for reporting because additional work needs to be done. I have not yet started looking at systems of ODEs having changing parameters. I do not yet have an idea about how to change the parameter.

All comments on these notes will be appreciated. Thank you in advance.

1.0 Introduction

I was struck by these statements at the bottom of Page 20 of the preprint mentioned in the Abstract:

"The Hurst or scaling behaviour has been found to be omnipresent in several long time series from hydrological, geophysical, technological and socio-economic processes. Thus, it seems that in real world processes this **behaviour is the rule rather than the exception**. The omnipresence can be explained based either on **dynamical systems with changing parameters** (Koutsoyiannis, 2005b) or on the principle of maximum entropy applied to stochastic processes at all time scales simultaneously (Koutsoyiannis, 2005a)." [My bolding]

I wanted to explore the presence of the behavior in ODE systems known to exhibit chaotic response. These systems of ODEs represent a source of chaotic response that is basic and additionally free of all the issues associated with measured data. The *phenomenology* of these systems has been adopted as the expected response from GCM codes. Exactly what that implies is not crystal clear. It seems, at least to zeroth-order, to mean that the sensitivity of the GCM calculations to initial conditions is an expected result, given that is also a fundamental property of the ODE systems. The enormous extrapolation from the clean systems of simple non-linear ODEs, for which numerical solution methods are both well-established and additionally simple to correctly and accurately code into software, to the GCMs is conveniently overlooked.

The characterization by Koutsoyiannis of "dynamical systems with changing parameters" also happens to fit with my previous discussions of the chaotic response of systems of non-linear ODEs. In the classic approach to these systems, the parameters are universally specified to be constant values. The corresponding quantities in GCMs, on the other hand, are constantly changing. The first investigations reported in these notes will focus on systems of ODEs having constant values for the parameters. Subsequent investigations might look into the case of changing values for the parameters.

Additionally, the discussions by Koutsoyiannis regarding fitting **assumed** trends to data series also struck a cord. The systems of non-linear ODEs which exhibit chaotic response cannot produce response having a trend. I wanted to see how this fact fit into the accepted wisdom of chaotic response and GCM results. The difficulties associated with assuming trends for measured data were introduced in the Abstract of the paper as follows:

"The most common modelling approach is to assume that long-term trends, which have been found to be omnipresent in long hydrological time series, are "deterministic" components of the time series and the processes represented by the time series are nonstationary. In this paper, it is maintained that this approach is contradictory in its rationale and even in the terminology it uses. As a result, it may imply misleading perception of phenomena and estimate of uncertainty."

Almost every discussion about the Global Mean Surface Temperature (GMST) includes assigning a linear trend to the recent measurements of the surface temperature. I wanted to see what various measured time-series for temperature look like relative to Hurst Coefficients. These investigations will be the subject of additional future notes.

The remainder of these notes are arranged as follows. Verification of the coding of the Hurst-coefficient methodology is briefly described in Section 2. Application of the method to calculations of chaotic response by the original Lorenz system of 1963 is given in Section 3.

2. Verification of the Code and Calculations

The equations needed to do the analysis have been given in the reference (see Page 19) and I won't repeat them here. Instead, I'll report the results of Verification exercises that I used to determine that the equations were correctly coded and were being correctly solved.

The methodology and equations for the Hurst-coefficient analysis were coded into a routine that returns the results when it is fed an array of values for the quantity of interest. In this way the same routine can be used in any arbitrary application and the coding remain unchanged.

Application of the method to numbers calculated by Lorenz-like systems of non-linear ODEs that exhibit chaotic response was an early objective. One of these equation systems was used as part of the Verification of the coding of the Hurst-coefficient method. The original Lorenz system of 1963 (XXXX need a reference) is among the most widely studied and it was used in these initial Verification activities.

The classic values of the parameters in the system; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, and the explicit Euler numerical solution method were used.

The first Verification calculations were done by forcing the numerical integration routine to return constant values; those constant values being the initial conditions (ICs) for the integration. This test is frequently termed a null transient and is very useful for finding incorrect coding. The explicit Euler method is especially simple in this regard; simply set y(i) = yold(i) (I=1, 2, 3) after initially setting yold(i) to the IC for each dependent variable. So as to make the Verification as simple as possible, numerical values of 1.0, -10.0, and 10.0 were set for the ICs. With these values some of the calculations to be compared with the code results are simply and directly obtained. Solutions for 100,000 steps of the integrations were generated and sent to the Hurst-coefficient routine. This test was successful, returning 0.00 for the standard deviations for all three equations for all lengths of all the series.

I next generated 1,000,000 random numbers and preformed several analyses of the results. The numbers were standardized to a mean value of 0.0 and standard deviation of 1.0. This provides a handy check on the coding of the calculations.

The first Hurst-coefficient analysis used 6 segment lengths (scales) (n = 1, 2, 4, 5, 8, 10) with the last segment having 100,000 entries. The results are shown in Figure 1 nearby. The logarithm of the standard deviation for each scale is shown as a function of the logarithm of the scale of the series. The left side of the plot represents the case of large numbers of short series (small number of entries in the series) and the right side represents the case of fewer and longer series.

The results of a power-law fit to the data points are also shown in the upper right-hand side of the Figure. The built-in fitting routine in Excel was used to determine the constant and the exponent for the fit. As shown in the Figure, a very good fit was obtained and the exponent, –0.4994, is very nearly the expected theoretical value of

–0.500 which gives the Hurst coefficient to be 0.500. The results in Figure 1 are in accord with the description of the process given by Koutsoyiannis, and provide Verification that the coding of the equations and their solution is correct.

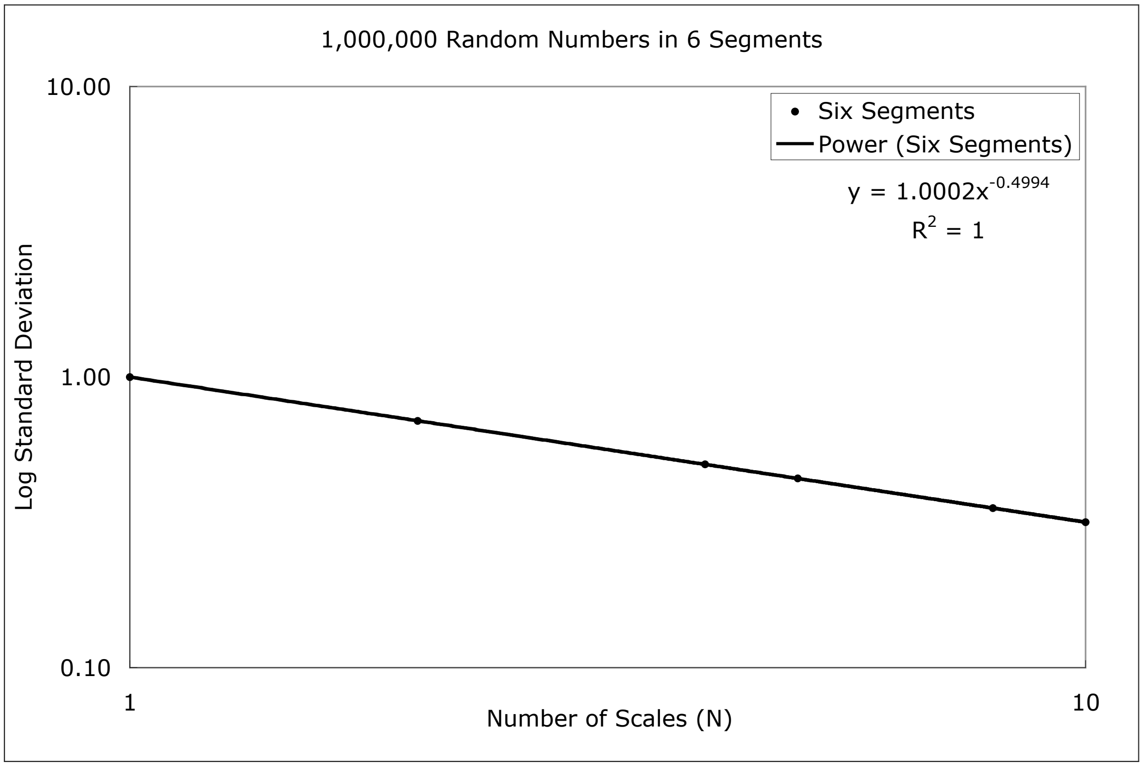

Figure 1. Standard deviations for one million random numbers.

The random numbers were also used for a few sensitivity studies looking at the effects of the number and length of the scales used in the analysis. The results of using 27 segment lengths, up to 500,000 entries in a segment, are shown in Figure 2 nearby. The 500,000 segment length means that only two segments cover the complete series. Using 1,000,000 for a segment length, of course, gives a nonsense number for the standard deviation and that result is not included on Figure 2. Note that as the scale of the segment increases the standard deviation begins to deviate from the theoretical line. Sill, the power-law fit to the numbers, shown in the upper right-hand corner of Figure 2, is not too bad.

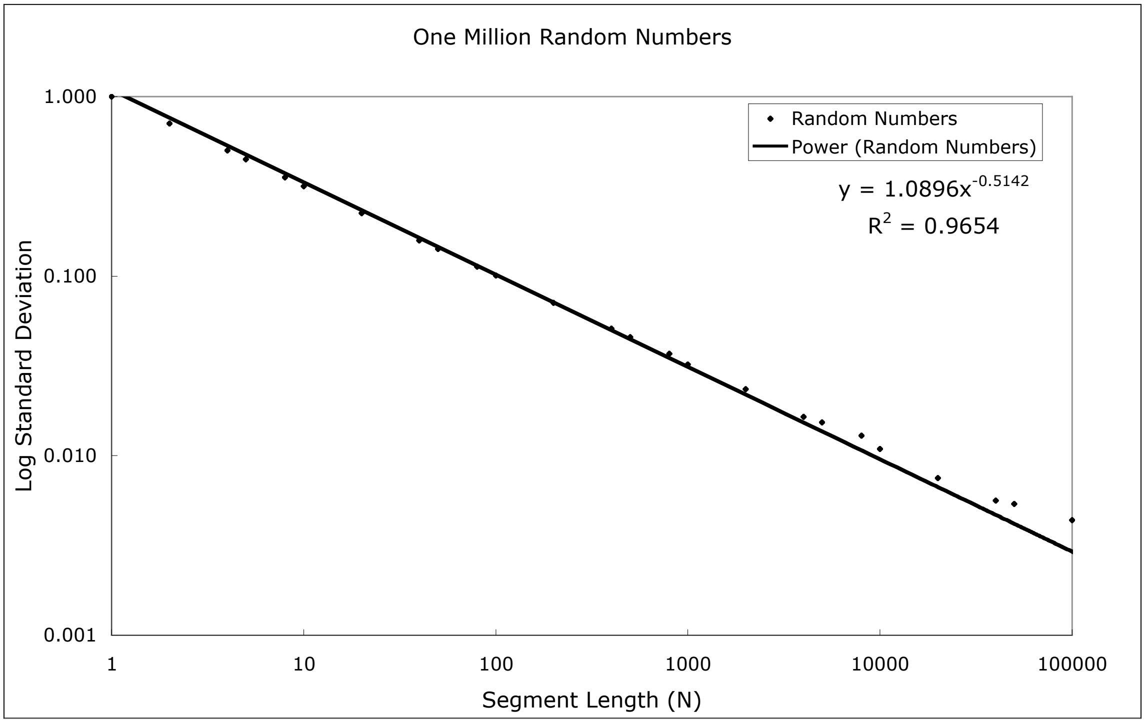

Figure 2. Standard deviations for one million random numbers using very long scales.

I also looked at the cases of two million and four million random numbers to check the effects of the density of numbers per scale. Using the same number of scales as for the original one million numbers increases the number density by two and four, respectively. The behavior was as in the original case with the largest scales, fewer very long segments, breaking up and deviating from the theoretical line. Still, the power-law fits were close to the expected theoretical value. The results for four million are shown in Figure 3.

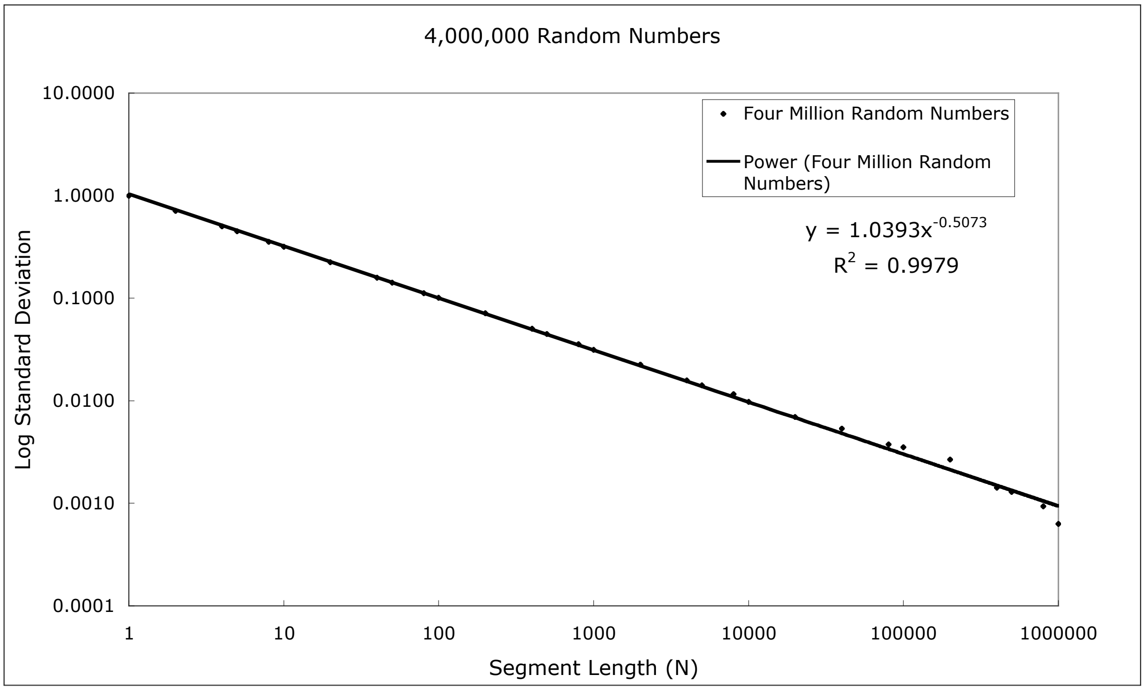

Figure 3. Hurst coefficient results for four million random numbers.

3. Application to the Original Lorenz System of 1963

So, now we want to see what chaotic response looks like in the Hurst-coefficient method as given by Koutsoyiannis. The reasoning is as follows. Chaotic response has been the hypothesis applied to temperature trajectories calculated with GCMs. That being the case, determining the Hurst-coefficient behavior of chaotic response as given by simple systems of ODEs might provide a signal to look for in calculated results from GCMs. Additional investigations are then needed to determine if measured temperature data exhibit Hurst-coefficient characteristics similar to (1) results from ODEs, and (2) results from GCMs. As noted previously, however, chaotic response cannot return results that have a trend associated with them. That property has been demonstrated an almost uncountable number of times in all investigations of chaotic system response.

3.1 The Lorenz Original System of 1963

The equation system devised by Lorenz, (Reference 1), is the standard relative to investigations of mathematical chaotic response. The equations are as follows:

$$\frac{dX}{dt} = -PrX + PrY$$
$$\frac{dY}{dt} = -Y + RaX - XZ \tag{1.1}$$
$$\frac{dZ}{dt} = -bZ + XY$$

The classic values of the parameters in the system; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, and the explicit Euler numerical solution method were used. The initial conditions (ICs) used for the first calculations were

$$(X, \quad Y, \quad Z) = (1.0, \quad -10.0, \quad 10.0)$$

The numerical solution was carried out to 100.0 Lorenz Time Units (LTUs) using a constant step size of 0.001, thus giving 100,000 entries in the time series.

Calculated results for X(t) are shown in Figure 4. There is nothing new there, and Y(t) and Z(t) show the same typical characteristics of aperiodic chaotic response. As shown in the Figure, and noted previously above, there is no trend with time associated with chaotic response.

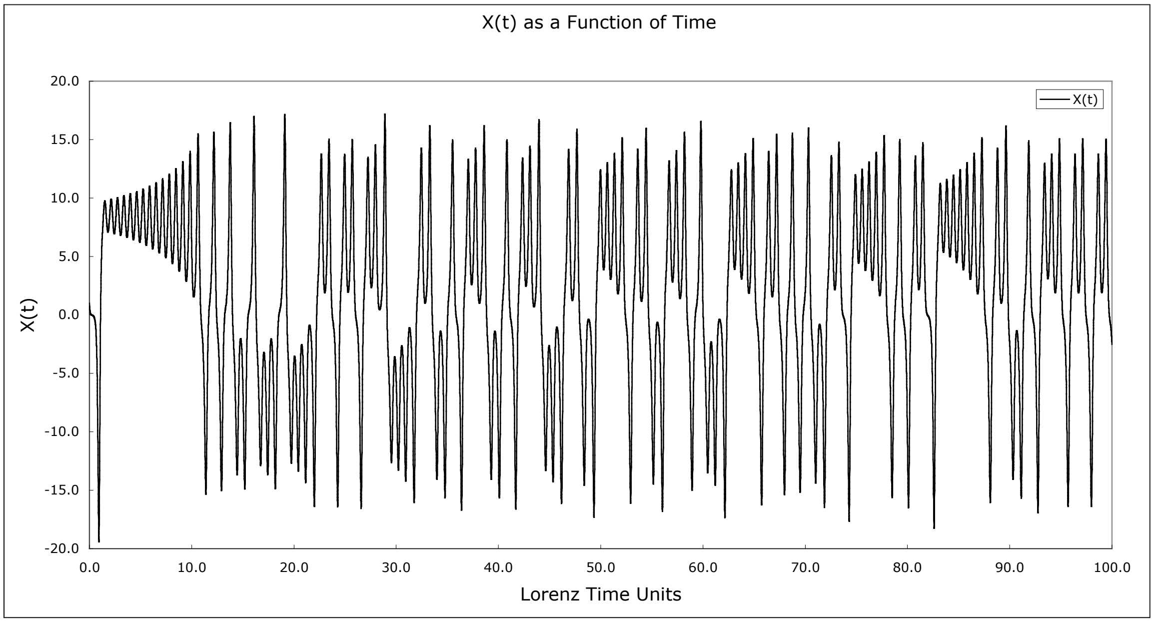

Figure 4. Typical characteristics for X(t) as a function of time; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, h = 0.001.

The results of the Hurst coefficient analysis is given in Figure 5. The results shown in Figure 5 are for the complete run time of 100,000 steps and for segment lengths from 1 to 25,000. Note that the latter value corresponds to just four segments to cover the entire time series. Now we see something interesting.

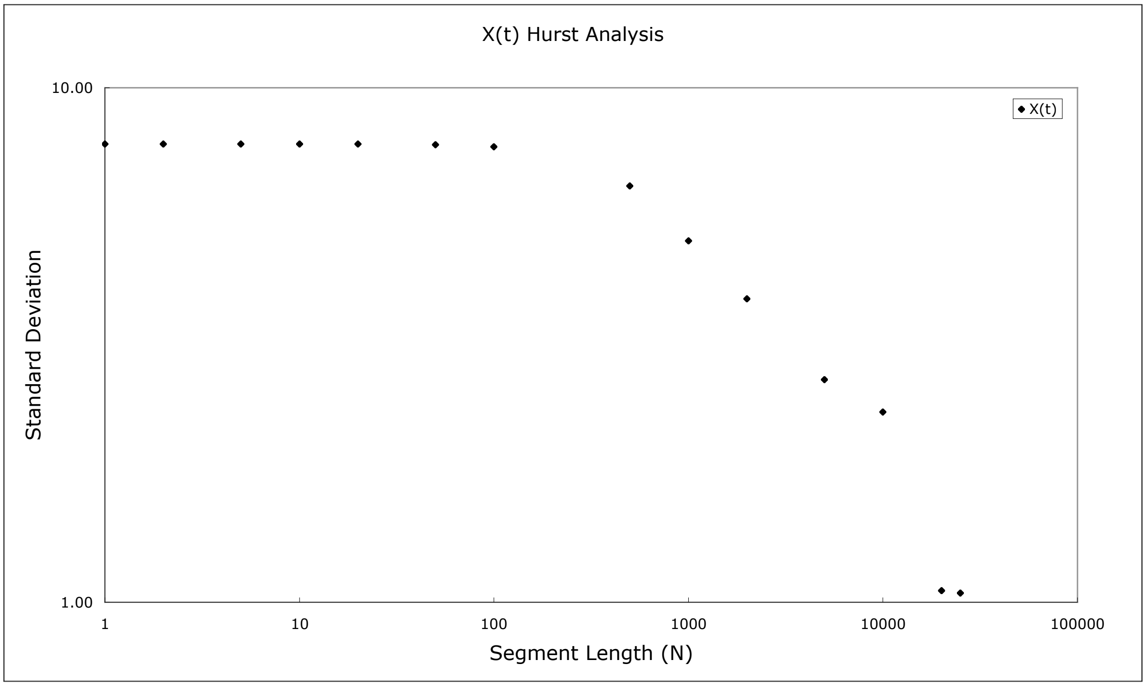

Figure 5. Hurst analysis plot for the complete Lorenz calculation; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, h = 0.001.

For segment lengths from 1 to about 500 the standard deviation is very nearly constant, at about 7.77 for this particular case. The Hurst coefficient for these segment lengths, all relatively short, is H = 1.0. We'll look at the properties of the longer segments below.

Many analyses of the Lorenz systems eliminate a "start-up" part of the results. Based on experience from previous analyses, about 20 LTUs is a good place to start. Re-doing the analysis, now having 80,000 entries, from 20.0 LTUs to 100.0 LTUs gives the results in Figure 6.

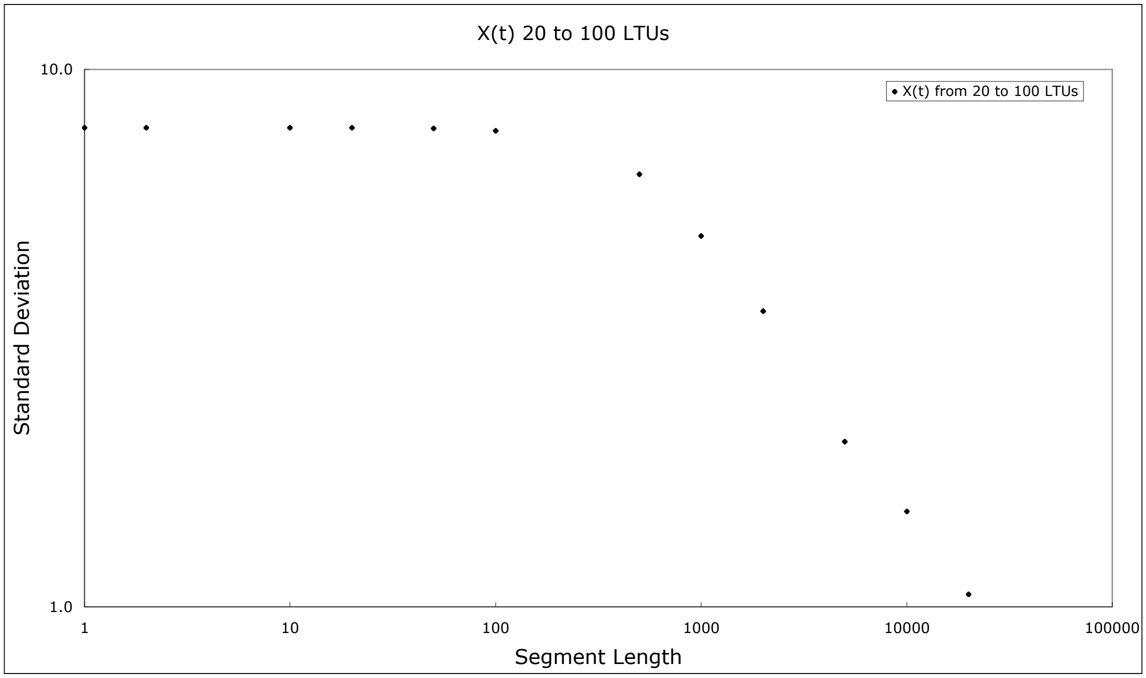

Figure 6. Hurst analysis plot for 20.0 to 100.0 LTUs; total length 80,000; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, h = 0.001.

Well, we see that the results look the same as those for the complete run shown in Figure 5 above. Not surprising, in my opinion. The Hurst Coefficient for the cases of shorter segment lengths is still H = 1.0. The constant value is now about 7.79.

If we now re-do the analysis, using the same parameter values, step size, and ICs, but carrying out the calculations to 1500.0 LTUs and omitting the first 500.0 LTUs from the Hurst analysis, we get the results in Figure 7. These are again for the X(t) dependent variable. Results for the other dependent variables all have the same characteristics.

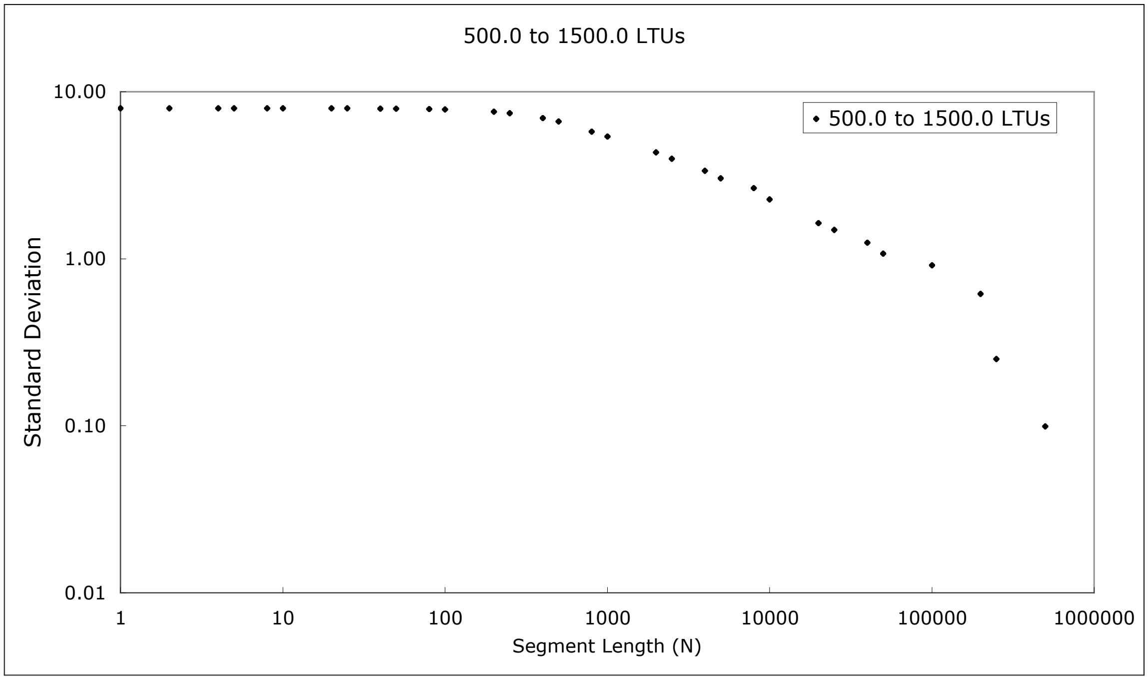

Figure 7. Hurst analysis plot for 500.0 to 1500.0 LTUs; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, h = 0.001.

And again the same characteristics are obtained; a Hurst Coefficient of H = 1.0 for the shorter segment lengths and then a break in the data at about segment length of 500. The last two data points correspond to segment lengths of four and two, respectively, and very likely are not valid.

Omitting parts of the calculated results from the analysis does not change the overall characteristics of the Hurst Coefficient. This result will not obtain if the solution goes to an equilibrium fixed point; such solutions are possible with the Lorenz 1963 system. In this case, the transient part of the calculation must be omitted. But then there's no problem left to analyze.

The definite break in the results at segment length of about 500 needs to be investigated. So, I plotted these for the complete run of Figure 5 and got Figure 8. Recall that the calculation for Figure 5 was carried out to 100.0 LTUs.

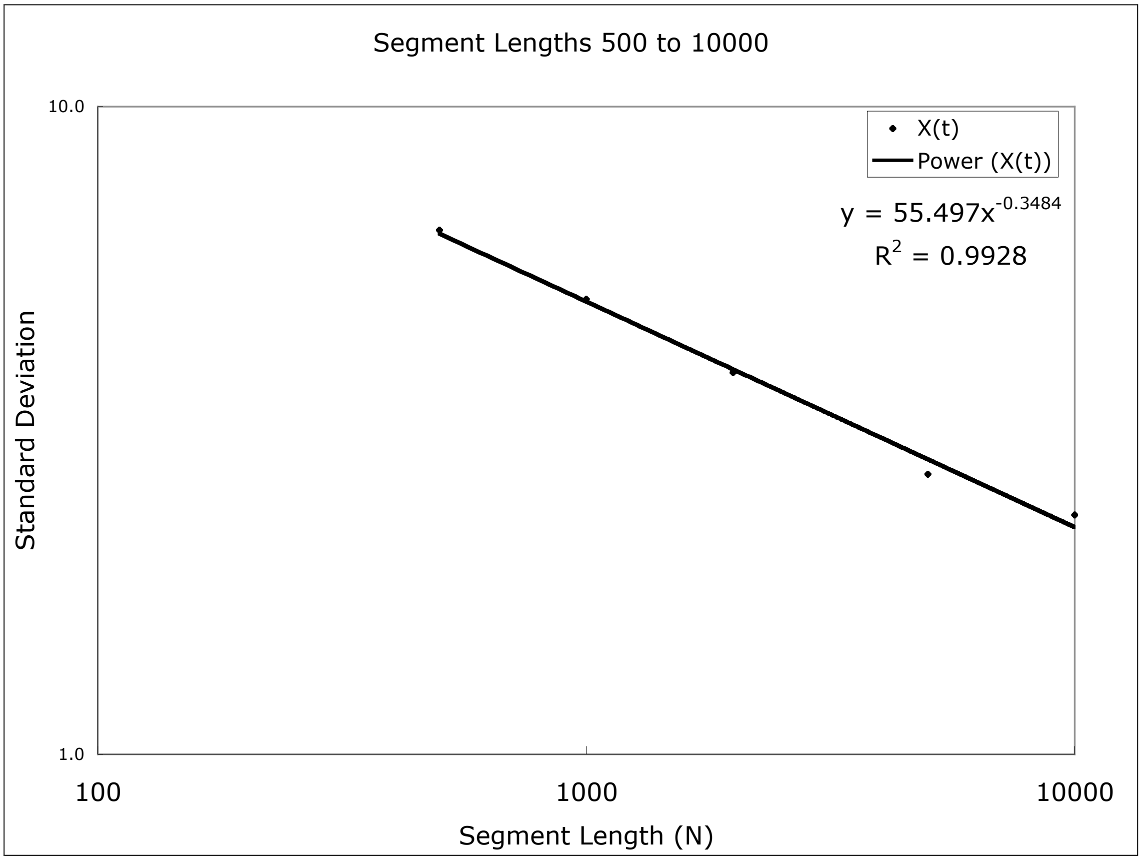

Figure 8. Hurst analysis for segment lengths 500 to 10,000 for the complete run of Figure 5; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, h = 0.001.

A power-law fit to the data points gave a slope of –0.3484; Hurst coefficient H = 0.652. The corresponding plot for the run from 20.0 to 100.0 LTUs shown in Figure 6 is shown in Figure 9.

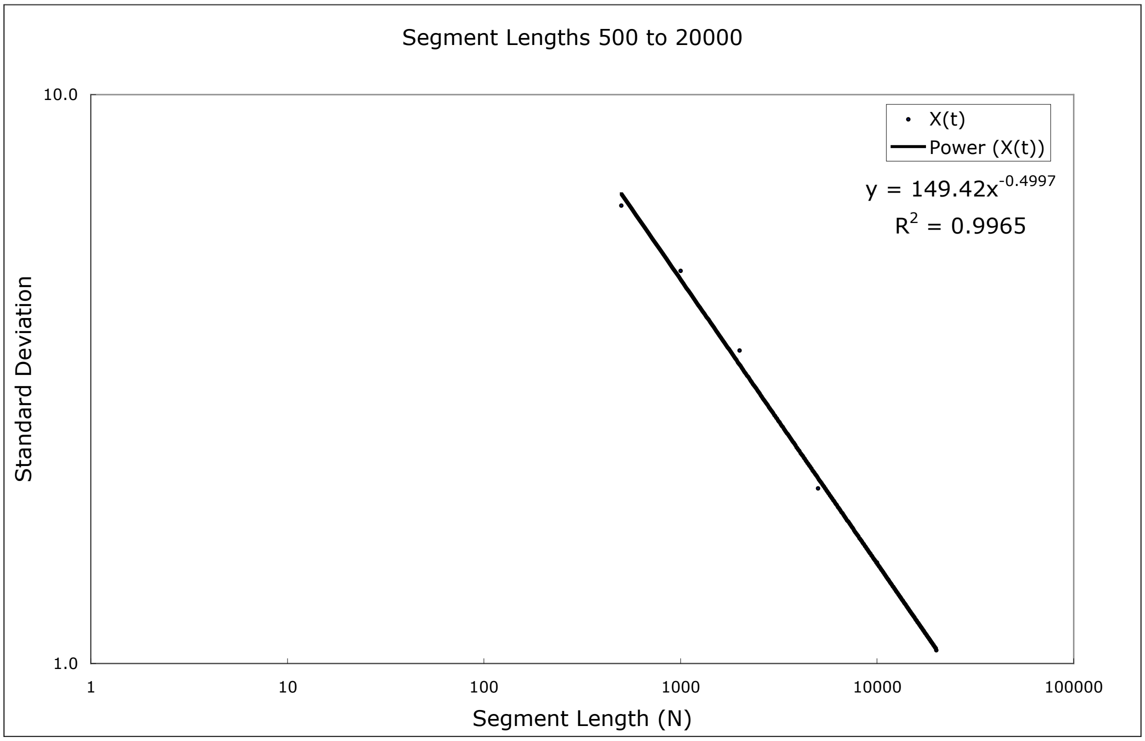

Figure 9. Hurst analysis for segment lengths 500 to 10,000 for 20 to 100 LTUs of Figure 6; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, h = 0.001.

The Hurst coefficient is now about 0.50, as in Figure 1 above in the case of random numbers. When using fewer and longer segment lengths, for this case, the statistics approach statistics for random numbers. I'm not yet sure what this means. It is clear from the results in Figures 5 through 8 that the over-all chaotic response is not random.

The corresponding results for the run of Figure 7, for which the first 500.0 LTUs were omitted from the analysis, gives the results in Figure 10. The last two data points have been omitted from Figure 10 and the associated power-law curve fit.

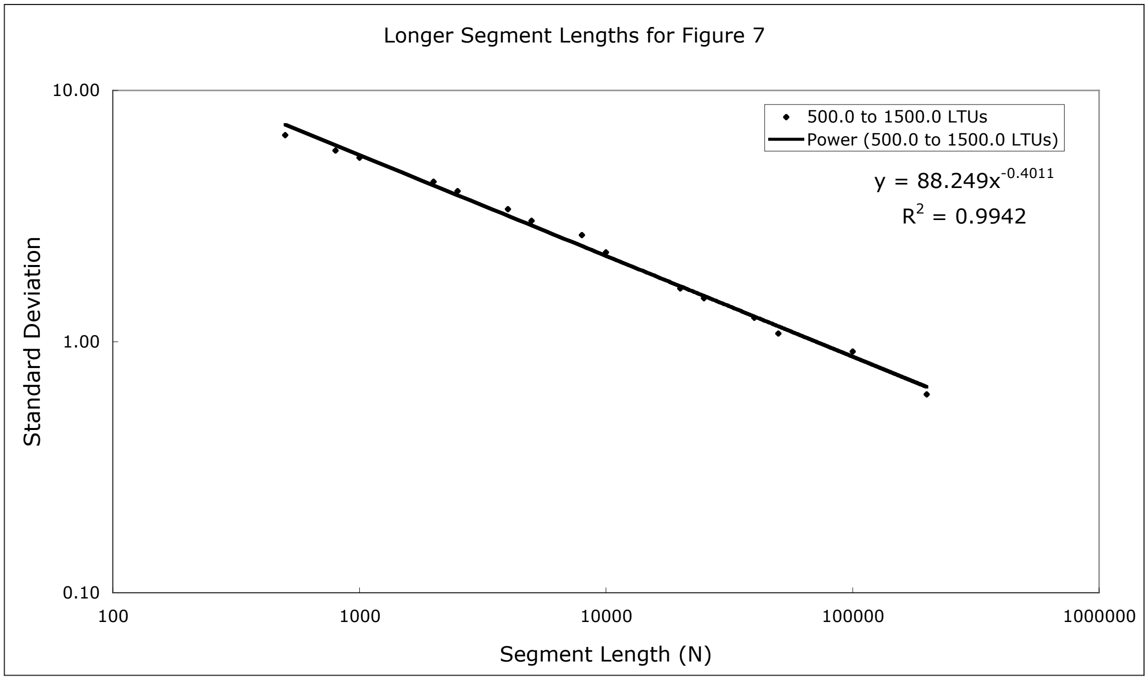

Figure 10. Hurst analysis for segment lengths 500 to 10,000 for 500.0 to 1500.0 LTUs of Figure 7; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, h = 0.001.

The Hurst Coefficient is now about H = 0.5989. If the last two data points are included in the power-law curve fit, the Hurst Coefficient comes to about H = 0.4838. Again, the Hurst Coefficients for longer but fewer segment lengths seem to approach that corresponding to random numbers.

Repeating the above analysis using anomalies, with the basis taken to be the complete series, gives the same results. Additionally, repeating the analyses using series normalized to zero average and unity standard deviation also give the same results. And, all preliminary first-looks at the analyses for Y(t) and Z(t) indicate those results will be the same as for X(t) given here.

3.1.1 Rough Initial Preliminary Summary and Conclusions

So long as the solutions for the system of equations exhibit chaotic response, that is an equilibrium fixed point is not attained, it is not necessary to omit a part of the series as a “start-up” process. The results indicate that the Hurst Coefficient for truly chaotic response when shorter segment lengths are used is H = 1.0.

A “start-up” processes very likely doesn’t obtain when natural phenomena and processes are the subjects of interest. Presumably, these natural phenomena have been operating for very long periods of time.

It is not yet clear what the characteristic associated with longer, but fewer, segment lengths means. The Hurst Coefficient can vary over a significant range for these segment lengths. In some cases the numerical value of the coefficient indicates that the data obtained from these numerical solutions are in fact random numbers.

3.2 Looking at Convergence of the Numerical Methods

As is well known, numerical solution methods applied to the ill-posed initial value problems represented by small systems of non-linear ODEs which exhibit chaotic response cannot be shown to converge as the step size is refined. This issue recently has been the subject of a couple of peer-reviewed papers (References 3 - 7).

While the lines of the trajectories in phase space cannot be shown to be independent of the step size, the general phase portrait does seem to approach a kind of convergence as the step size is refined; the over-all phase diagram changes less and less as the step size is refined. Effectively, each step size gives a different algebraic map of the continuous equations to the discrete approximations. Different numerical solution methods will also give different solutions.

The calculations in the previous discussion above were repeated using a larger step size of h = 0.01 LTUs and the calculation carried out to 10,000 LTUs. Convergence is not the subject here for this calculation. Instead this is a zeroth-order cut at looking into the effects of the step size.

The results for X(t), normalized, are shown in Figure 11 for the over-all view and Figure 12 for the longer-segment view. Note that for this step size, the break in the plot occurs at segment lengths of about 50. In contrast to the previous cases for which the break occurred at about 500.

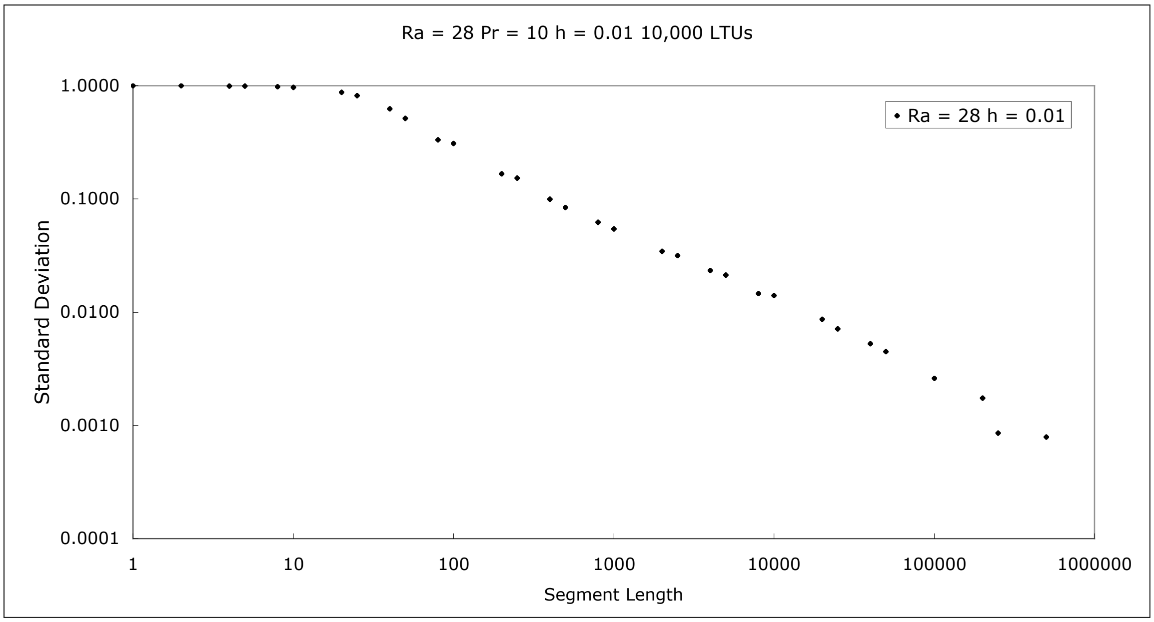

Figure 11. Hurst analysis for all segment lengths for a calculation out to 10,000 LTUs; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, h = 0.01.

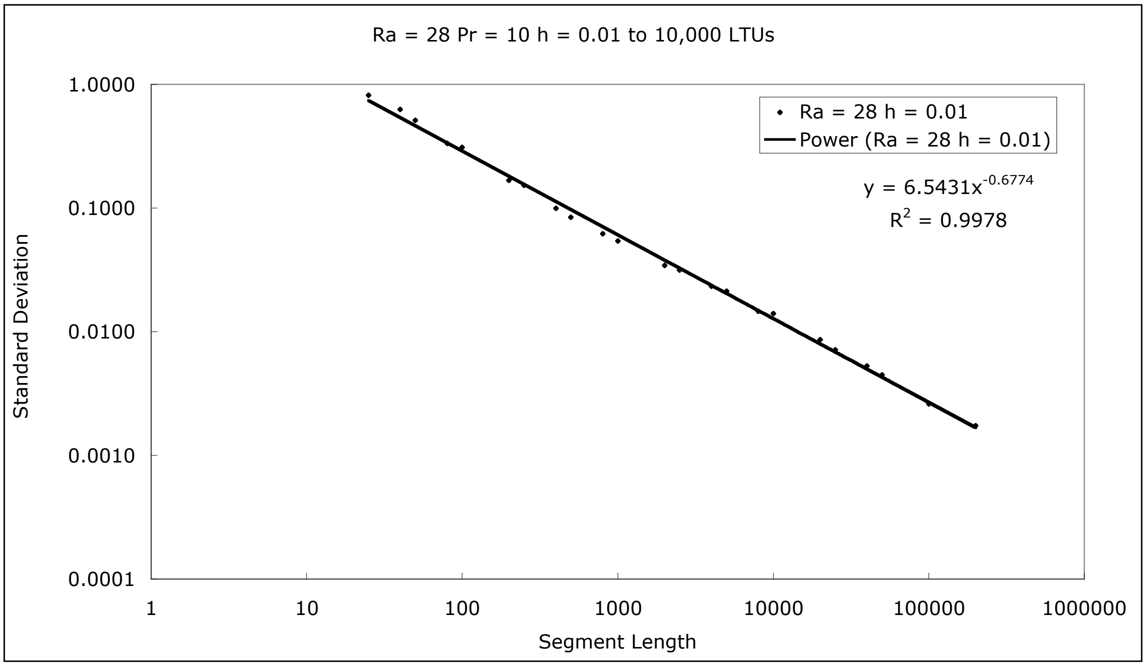

Figure 12. Hurst analysis for segment lengths from 25 to 200,000 for the calculation of Figure 11; *Ra* = 28.0, *Pr* = 10.0, and *b*=8.0/3.0, h = 0.01.

The Hurst Coefficient for the longer segment lengths given in Figure 12 is about H = 0.3226.

So far we've seen Hurst Coefficients, when longer segments are the focus of the analysis, ranging from about 0.32 to about 0.65. When the Hurst-Coefficient analysis is based on the shorter segment lengths, the Hurst coefficient is H = 1.0.

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