Transient Compressible Method of Exact Solutions Verification Problems
Dan Hughes
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Abstract

Analytical solutions for the pressure response in a compressible fluid following an instantaneous stopping of the flow are obtained. A zeroth-order cut at accounting for fluid-structure interaction is included in the model equations. Sudden closure of a valve is an example. The fluid speed response and the change in the volume of the fluid container are obtained by use of the pressure response. For the case of two flow channels coupled through a common flexible wall, the basic analysis method continues to hold, but numerical solution methods are needed.

The solutions are candidates for application as Method of Exact Solutions (MES) verification problems. As for the case of almost all MES problems, the terms in the model equations that are tested are limited. The flows considered here test the correctness of handling boundary conditions and the application of general models and methods to extremely simple geometries. The candidates also avoid some of the difficulties associated with using shock-tube like problems for verification of transient, compressible flow models and solution methods that use finite difference equations and not the method of characteristics. I am not aware that solutions for shock-tube like problems with fluid-structure interactions are available.

These notes will summarize the case of mechanical interaction between the fluid and the container wall. Other notes will address the case of thermal interactions.

Introduction

The flow-channel geometry consists of a single mass and energy cell, closed at one end, with an associated momentum cell at the other end. The nomenclature is related to the universal staggered-grid approach used in finite-difference approximations to the continuous form of the basic fluid-flow equations. In many applications the mass and energy cell, to which the scalar mass and energy equations are applied, are labeled nodes or control volumes among other designations. The associated momentum cells, to which the components of the momentum balance model equations are applied, are sometimes referred to as cell-edges, links, or junctions. The staggered grid was an important aspect of the numerical solution methods developed at the Los Alamos Scientific Laboratory, now the Los Alamos National Laboratory, by Frank Harlow and colleagues for the Particle in Cell, PIC, solution method.

For these notes, whenever necessary, a subscript '*k*' will usually be associated with the nodes and a subscript '*j*' associated with the links. For the analytical solutions, the subscripts will not be needed. I have attempted to show a sketch in the nearby Figure

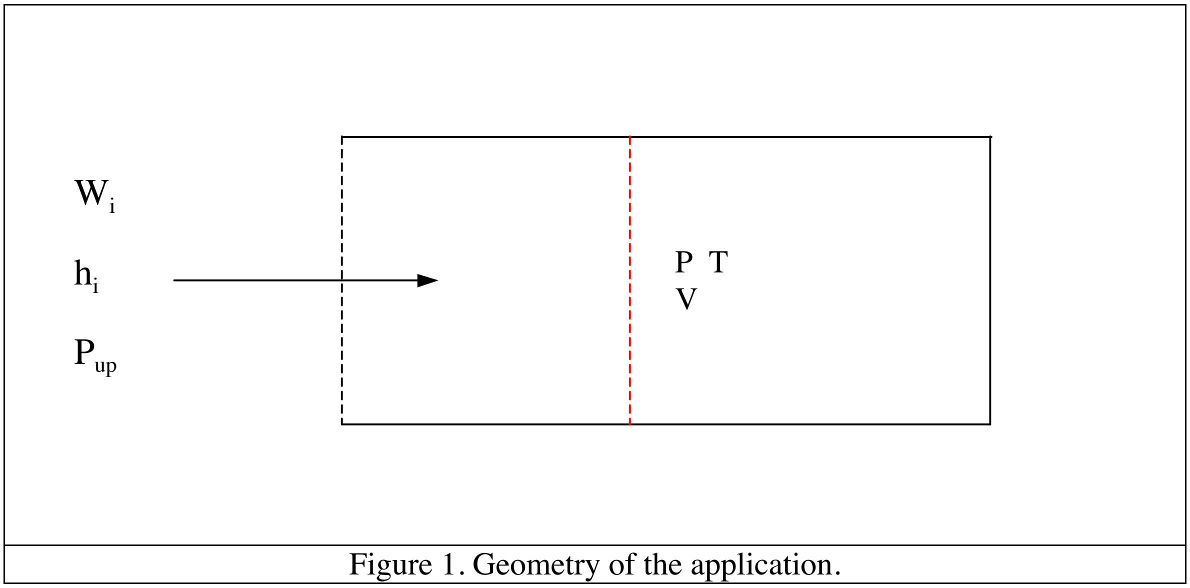

Figure 1. Geometry of the application.

The mass and energy node is bounded by the solid lines and the black dashed line at the left-hand edge. The red dashed lines marks the center of the mass and energy node and the right-hand edge of the momentum link. Generally, solution variables associated with mass and energy are assigned to be located at the center of the mass and energy node; the thermodynamic equation of state quantities, pressure for example, are among these. Relative to the momentum balance model equation, this problem is almost all boundary conditions. The numerical solution methods, however, will have to consider that only half a momentum link is present in these flows. And that if the fluid flow rate at the inlet is not specified and boundary link approach will be needed in order to calculate that flow rate.

For the case of two nodes mechanically coupled through a flexible wall, the picture in Figure 1 has another node adjacent and parallel to the node shown there. One of the nodes will be labeled '*k*' and the other labeled '*z*'.

This application procedure is more often associated with one-dimensional process-model approaches than for multi-dimensional CFD-like applications. However, some CFD-like codes might be able to accommodate use of a single closed-end node. Process-model like approaches generally have great flexibility relative to description of the channel geometry and flexible control over the empirical correlations, friction factor and heat transfer coefficient for examples, that enter a calculation. For the present applications, rigorous correspondence with the analytical solutions will require this flexibility. Additionally, specification of initial conditions that do not satisfy the basic equation models is also generally possible with these types of codes.

Well, that a lot of stuff for just introducing the flow-channel geometry.

The Model Equations

A mass conservation model for the flow, applied to the node is

$$\frac{dM}{dt} = W_i\left(t\right) \tag{1.1}$$

where M is the mass of fluid in the pipe,

$$M = \rho V \tag{1.2}$$

and $W_i\left(t\right)$ is the mass flow rate of fluid into the pipe at the upstream boundary

$$W_i\left(t\right) = \rho U A_f \tag{1.3}$$

where ρ is the fluid density, U is the fluid speed at the entrance of the pipe, and A_f is the flow area for the pipe. The initial condition is

$$M(0) = M_0 \tag{1.4}$$

An energy balance model equation, also applied to the node, is

$$\frac{dE}{dt} = h_i W_i\left(t\right) \tag{1.5}$$

where h_i is the enthalpy of the fluid entering the pipe at the upstream boundary and the energy content for the fluid is

$$E = Mu = \rho V u \tag{1.6}$$

where u is the specific internal energy for the fluid. The initial condition is

$$E(0) = E_0 \tag{1.7}$$

The pressure-volume work term has been omitted from the energy equation model. Calculations following the solution can be used to show that the term is small and neglecting it is justified.

Neglecting all terms in a momentum balance model except for the pressure gradient, a momentum equation for the fluid flow into the pipe at the upstream end is

$$\frac{dW_i}{dt} = \frac{A_f}{L}\left(P_{up} - P\right) \tag{1.8}$$

where P_{up} is the constant upstream pressure and L is a representative distance for the pressure gradient; one-half the pipe length. The initial condition is

$$W_i(0) = W_i^0 \tag{1.9}$$

Note that wall friction and gravity have not been included. Gravity is easily included, but wall friction is not. For verification purposes, the wall friction can be minimized by use of a few different techniques available in most models and codes; make the pipe diameter be really big, for example.

The pressure for the fluid in the node for the single-node case is given by the equation of state

$$P = \hat{P}(M, E) \tag{1.10}$$

and likewise for all other thermodynamic state quantities. The thermodynamic state of only a single node is required so long as the response of that node is not related to the response of a node which is coupled through the container wall.

The volume of the node is given by a force balance for the wall, the wall function. For the case of constant pressure on the outside of the pipe, a force balance gives

$$V = V_0 + K_w\left(P(t) - P(0)\right) \tag{1.11}$$

where K_w is the wall compliance, and subscript ’0’ refers to the initial value. Specific forms for K_w are available for various types of pipe supports (boundary conditions). Generally for the calculations in this note, pure radial expansion of the pipe wall is assumed. For this condition

$$K_w = \frac{\pi}{4} D^2 L \left(\frac{1-\mu^2}{E}\right)\left(\frac{D}{\bar{t}}\right) \tag{1.12}$$

For the case of a varying pressure on the outside of the pipe wall, we use the following nomenclature. Let subscript ’*k*’ refer to the volume of interest, and ’*z*’ refer to a fluid volume on the other side of the wall, the wall function is then

$$V_k = V_{k0} + K_w[\left(P_k(t) - P_k(0)\right) - \left(P_z(t) - P_z(0)\right)] \tag{1.13}$$

For the coupled-node case, the EOS for the pressure for node ‘*k*’ is

$$P_k = \hat{P}_k(M_k, E_k, M_z, E_z) \tag{1.14}$$

with a similar expression for node '*z*'.

These equations give a complete specification for the problems.

Solutions

Two analytical solutions will be developed in the following discussions; (1) the flow at the inlet is specified as a function of time and the pressure response is calculated, and (2) the initial flow is specified and the subsequent flow and pressure response are calculated. Solutions of both rigid- and flexible-wall containers are developed.

The key to the solutions is through the time rate of change of the pressure equation of state. For the single-node case, Eq. (1.10) gives

$$\frac{dP}{dt} = \left(\frac{\partial P}{\partial M}\right)_E \frac{dM}{dt} + \left(\frac{\partial P}{\partial E}\right)_M \frac{dE}{dt} \tag{1.15}$$

The pressure history for the pipe is given by putting Eqs. (1.1) and (1.5) into Eq. (1.15) to get

$$\frac{dP}{dt} = \left[\left(\frac{\partial P}{\partial M}\right)_E + h_i \left(\frac{\partial P}{\partial E}\right)_M\right] W_i(t) \tag{1.16}$$

The derivatives of the intensive thermodynamic property, P, with respect to the extensive macro properties M and E are obtained by use of implicit function theory. The method and several results have been given at this URL link:

http://models-methods-software.com/2011/01/14/implicit-function-theory-applications-part-0/

For the case of a rigid pipe wall, the results are

$$\left(\frac{\partial P}{\partial M}\right)_E = \frac{C_{sf}^2}{V}\left(1 - \frac{h\beta}{C_p}\right)$$

and (1.17)

$$\left(\frac{\partial P}{\partial E}\right)_M = \frac{C_{sf}^2}{V}\frac{\beta}{C_p}$$

where C_{sf}^2 is the sound speed for the fluid

$$C_{sf}^2 = \frac{C_p \upsilon}{C_v \kappa} \tag{1.18}$$

where υ is the specific volume for the fluid, C_p is the constant-pressure specific heat, C_v is the constant volume specific heat, β, is the coefficient of thermal expansion and κ is coefficient of isothermal compressibility.

Putting Eqs. (1.17) into Eq.(1.16) will give

$$\frac{dP}{dt} = \frac{C_{sf}^2}{V} W_i(t) \tag{1.19}$$

For the case of a constant pressure on the outside of a flexible-wall pipe, the volume *V* is a function of the pressure *P* in the pipe as given by the wall function, Eq. (1.11). For this case, implicit function theory gives the derivatives to be

$$\left(\frac{\partial P}{\partial M}\right)_E = \frac{C_{eff}^2}{V}\left(1 - \frac{h\beta}{C_p}\right)$$

and (1.20)

$$\left(\frac{\partial P}{\partial E}\right)_M = \frac{C_{eff}^2}{V}\frac{\beta}{C_p}$$

where the effective pressure-wave speed is

$$\left(\frac{V}{C_{eff}}\right)^2 = \frac{M^2 \upsilon^2}{C_{sf}^2} + MK_w\left(1 - P\frac{\upsilon\beta}{C_p}\right) \tag{1.21}$$

where $K_w = \left(dV/dP\right)$ is given by Eq. (1.12). Putting Eqs. (1.20) into Eq. (1.16) gives

$$\frac{dP}{dt} = \left(\frac{V}{C_{eff}}\right)^2 W_i(t) \tag{1.22}$$

If the volume 'z' also deforms the volume in each pipe is a function of both P_k and P_z and the volume is given by the wall function of Eq. (1.13). This more general case will be discussed below later in these notes.

Inlet Fluid Flow Rate Specified

The solutions for the case that the fluid flow rate at the inlet is specified is straightforward. These are given for the rigid-wall and deformable-wall case.

Rigid Wall Pipe

If the fluid flow rate at the pipe inlet is specified as a function of time, the pressure response is obtained by directly integrating the corresponding equation for the pressure. If

$$W_i(t) = A_w - B_w t \tag{1.23}$$

for example, then Eq. (1.19) gives

$$P(t) = P_0 + \frac{C_{sf}^2}{V_0}\left(A_w t - \frac{1}{2} B_w t^2 \right) \tag{1.24}$$

if all the quantities in the equations can be assumed to be constant. Comparisons of calculations with codes for which variations are not assumed to be small or zero, indicates that this is an excellent assumption.

The mass of fluid and energy content as a function of time are obtained by integrating Eq. (1.1) and Eq. (1.5), respectively.

The temperature response is obtained in the same manner as the pressure response from

$$\frac{dT}{dt} = \left[\left(\frac{\partial T}{\partial M} \right)_E + h_i \left(\frac{\partial T}{\partial E} \right)_M \right] W_i(t) \tag{1.25}$$

Given the mass and energy content, the specific internal energy is

$$u = \frac{E}{M} \tag{1.26}$$

and the enthalpy is

$$h = \frac{E}{M} + PV \tag{1.27}$$

For the rigid-wall case, the volume, of course, is constant.

Flexible (Deformable) Wall Pipe

All that changes is that the effective sound speed of Eq. (1.21) is used in the equations given in the Section just above. The volume is obtained from Eq. (1.11) with the pressure response substituted for $P(t)$.

Calculated Results

A sample calculation has been carried out using the time-dependent specified fluid mass flow arte at the inlet of a pipe. The total pipe length is 10.0 m and the flow area is 1.0 m^2 . The fluid was taken to be subcooled liquid water at an initial pressure of 0.20 MPa and temperature 350.0 K. The fluid enthalpy at the inlet and initial specific internal energy correspond to these conditions. The pipe-wall is taken to be both rigid and flexible (not at the same time), but not super stiff so that the effective pressure-wave speed of Eq. (1.21) is $C_{sf} = 642.3$ m/s . The sound speed for the subcooled liquid water is about 1540.0 m/s, and that's the value used in the analytical solution for the rigid-wall case.

The pressure response for a rigid wall and a deformable wall are shown in the nearby Figure 1.

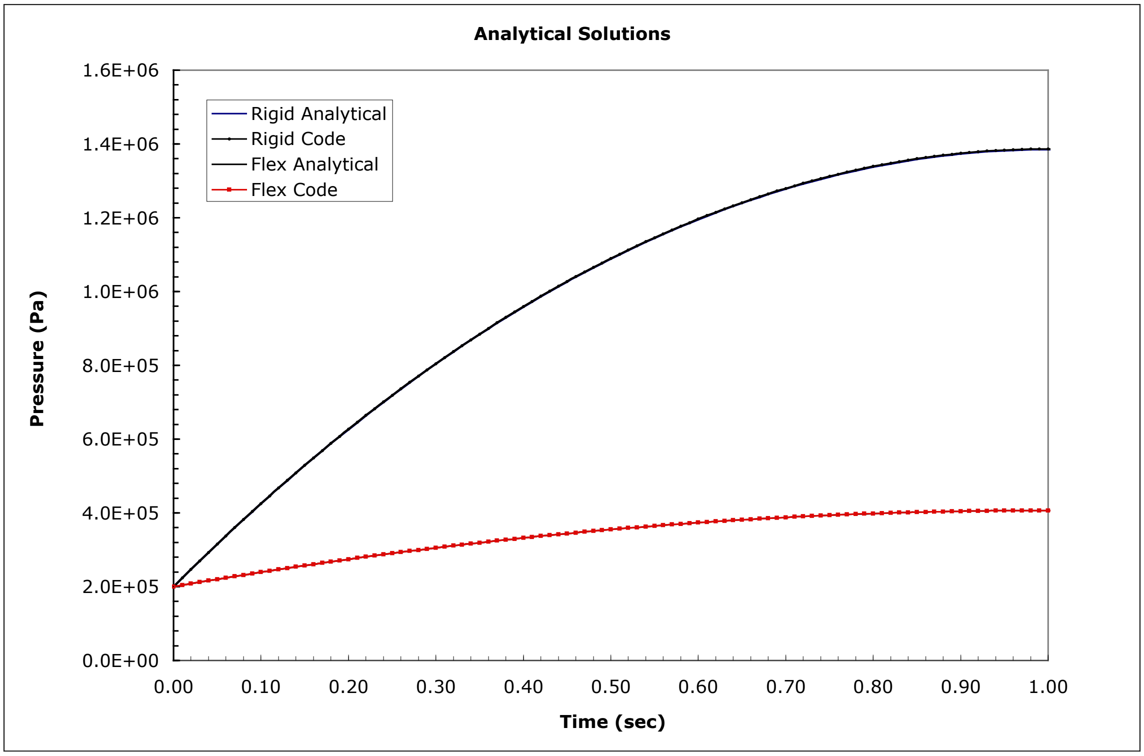

Figure 1. Pressure response for specified inlet flow case.

As shown in Figure 1, the pressure increase is significantly greater for the rigid-wall case then for the flexible-wall case.

The change on the volume for the 1.0kg of mass added into the pipe is shown in the nearby Figure 2.

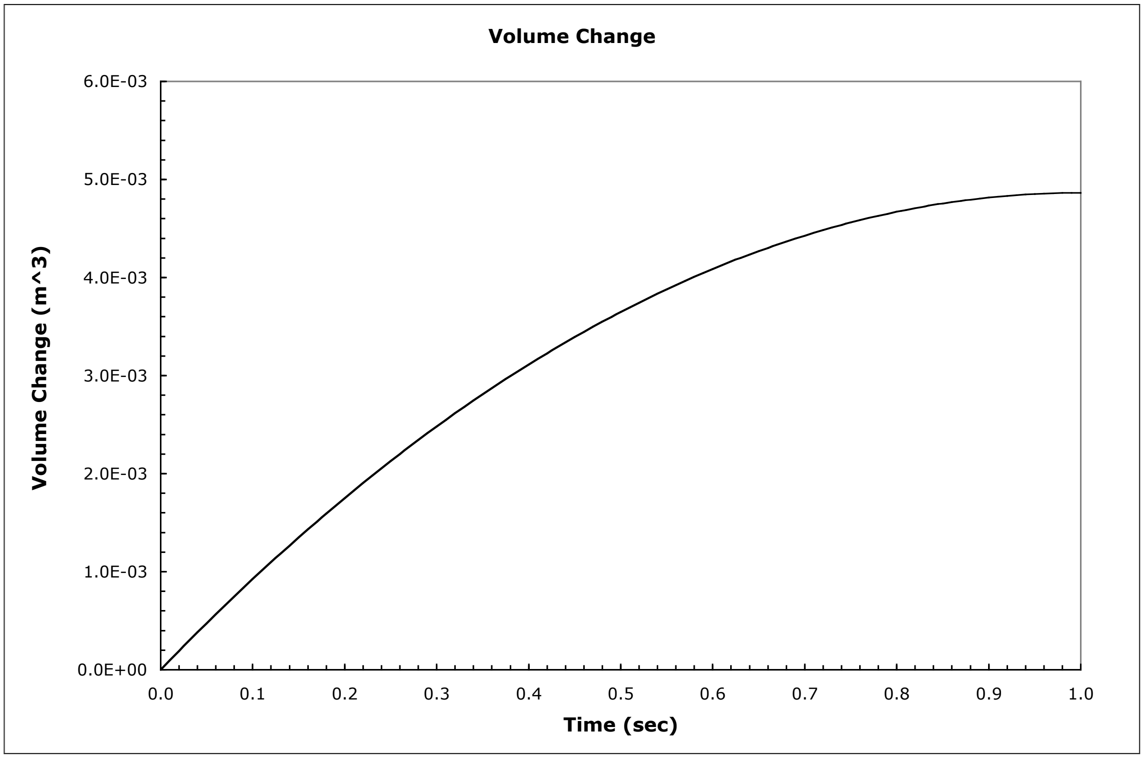

Figure 2. Volume change as a function of time for specified flow case.

Initial Flow and Pressure Distribution Specified

This problem can be viewed in a couple of different ways; as an actual steady flow along a pipe for which a valve is instantaneously shut, or (not completely) arbitrary values for the initial conditions, mass flow rate at the inlet, pressure at the inlet, and pressure at the center of the mass and energy node, can be specified and the response calculated; flow against an already-closed valve, perhaps.

The nature of the analytical solution for the pressure response will be illustrated by use of the rigid-wall case. Taking the derivative of Eq. (1.19) and putting the momentum equation model of Eq. (1.8) into that result gives

$$\frac{d^2P}{dt^2} = \frac{C_{sf}^2}{V}\frac{A_f}{L}\left(P_{up} - P\right) \tag{1.28}$$

The equation needs two initial conditions

$$P(0) = P_0$$

and (1.29)

$$\left.\frac{dP}{dt}\right|_{t=0} = \frac{C_{sf}^2}{V_0} W_i(0)$$

The solution that satisfies the boundary conditions if the flow-channel geometry is such that $V = A_f L$, as it usually is,

$$P(t) = P_{up} + \left(P_0 - P_{up}\right)\cos\left(C_{sf}\frac{\sqrt{2}}{L}t\right) + C_{sf}W_0\frac{1}{\sqrt{2}}\frac{1}{A_f}\sin\left(C_{sf}\frac{\sqrt{2}}{L}t\right) \tag{1.30}$$

where it has again been assumed that everything is constant, even the volume of the container. Note that wall friction has not been included, and gravity hasn't been either. Gravity is easily included, but wall friction is not. For verification purposes, the wall friction can be minimized by use of a few different techniques; setting the pipe diameter to be a really big number, for example. An approximate solution for this case is then

$$P(t) = P_0 + C_{sf}W_0\frac{1}{\sqrt{2}}\frac{1}{A_f}\sin\left(C_{sf}\frac{\sqrt{2}}{L}t\right) \tag{1.31}$$

For the case of a flexible wall with constant pressure on the outside, the same solution obtains with C_{eff} taking the place of C_{sf} . The change in the volume, or mass, is assumed to be sufficiency small that it can be neglected.

Equation (1.31) can be used to get the maximum pressure and its time of occurrence. The maximum pressure occurs at

$$t_{P\max} = \frac{\pi}{2\sqrt{2}}\frac{L}{C_{sf}} \tag{1.32}$$

at which time the fluid speed has reached zero prior to reversing direction, and the maximum pressure is

$$P_{\max} = P_0 + C_{sf}W_0\frac{1}{\sqrt{2}}\frac{1}{A_f} \tag{1.33}$$

Note that the maximum pressure increase is different from the usual incompressible-fluid case

$$\Delta P_{\max} = C_{sf}\Delta\frac{W}{A_f} = \frac{1}{2}\rho C_{sf}\Delta U \tag{1.34}$$

where the change in the fluid speed is usually taken to be the initial speed and the final speed is considered to be zero. The maximum pressure for the compressible case for the geometry considered here is greater by a factor of $\sqrt{2}$ than the incompressible case.

Following the solution for the pressure history, the mass flow rate history, and thus the fluid speed, can be obtained by straightforward analytical integration of Eq. (1.8) using the pressure response of Eq.(1.30).

The approximate, but excellent, solution of Eq. (1.31) gives

$$U(t) = \frac{W_0}{\rho A_f} \cos\left(C_{sf} \frac{\sqrt{2}}{L} t \right) \tag{1.35}$$

The mass content is obtained from Eq. (1.1) using Eq. (1.35) or the more exact solution of Eq. (1.30) and energy content is obtained from Eq. (1.5), again using Eq. (1.35) or the more exact solution of Eq. (1.30).

Finally, the volume of the pipe as a function of time is obtained by putting the pressure history into Eq. (1.11).

Calculated Results

The model has been applied to a straight round pipe having constant cross-sectional flow area. The total pipe length is 10.0 m and the flow area is 1.0 m^2. Specified pressure boundary conditions at the inlet and outlet of the pipe were used to establish the steady initial flow in the channel. The fluid was taken to be subcooled liquid water at a pressure of 0.20 MPa and temperature 350.0 K. The water being nearly incompressible and the fluid speed essentially constant along the flow, the pressure distribution is basically linear along the pipe and the initial pressure at the center of the flow channel is basically the average of the specified upstream and downstream values. The level of the initial fluid speed is set by the specifications of the pressure-gradient properties of the pipe wall and pressure drop between the inlet and outlet. For the conditions specified for this application, the initial fluid speed is about 5.686 m/s.

Following establishment of the initial steady state, the flow at the downstream end is instantaneously stopped. The pipe-wall is taken to be flexible, but relatively stiff so that the effective pressure-wave speed is $C_{sf} = 1103.0 \mathrm{m/s}$. The sound speed for the liquid water is about 1540.0 m/s.

I have access to a model and code that have flexible-volume capabilities. The following calculations were carried out with that model / code. Additionally, I have written a standalone code that solves the ODEs outlined in these notes. Some of the variable-volume calculations can be carried out using a spreadsheet, and I have used that approach also.

The pressure history at the center of the pipe is shown in Figure 3, and the fluid speed is shown in Figure 4. Note that if the calculation is carried beyond the end time shown in the Figures, negative values for the pressure can be calculated by the analytical solution.

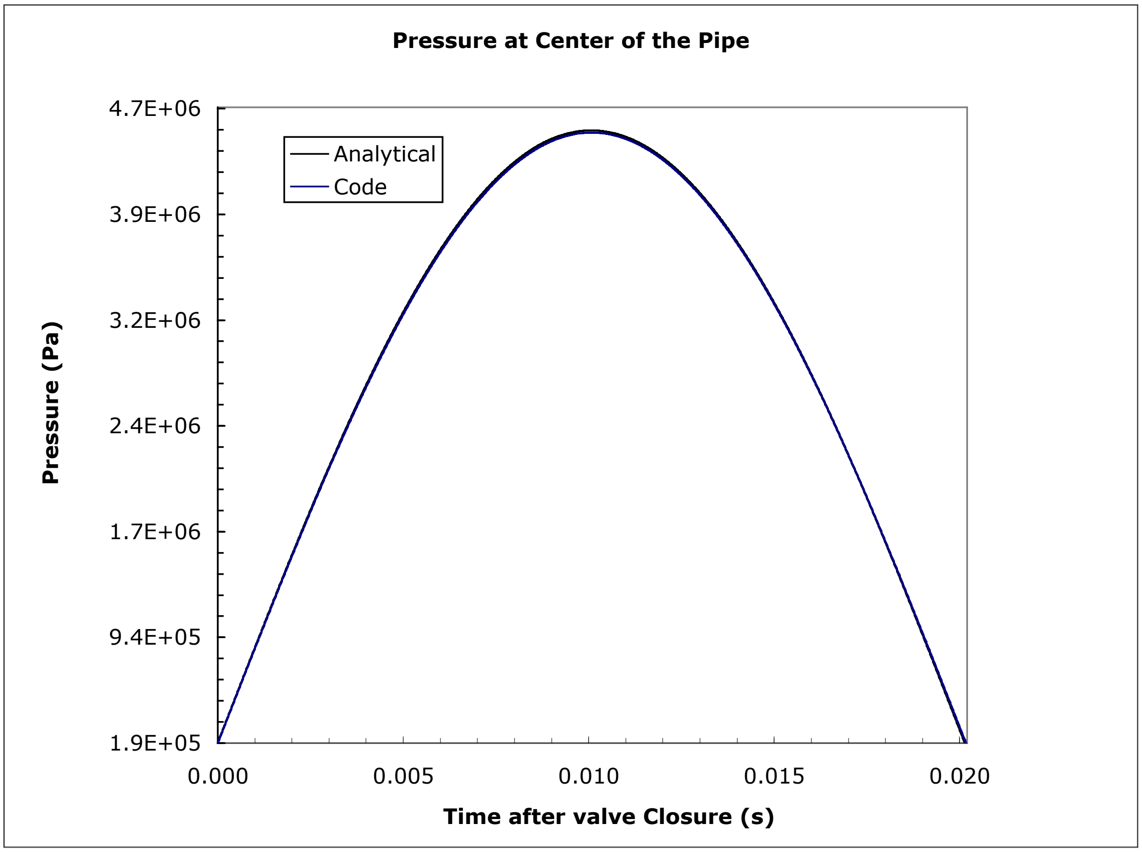

Figure 3. Pressure response following instantaneous valve closure.

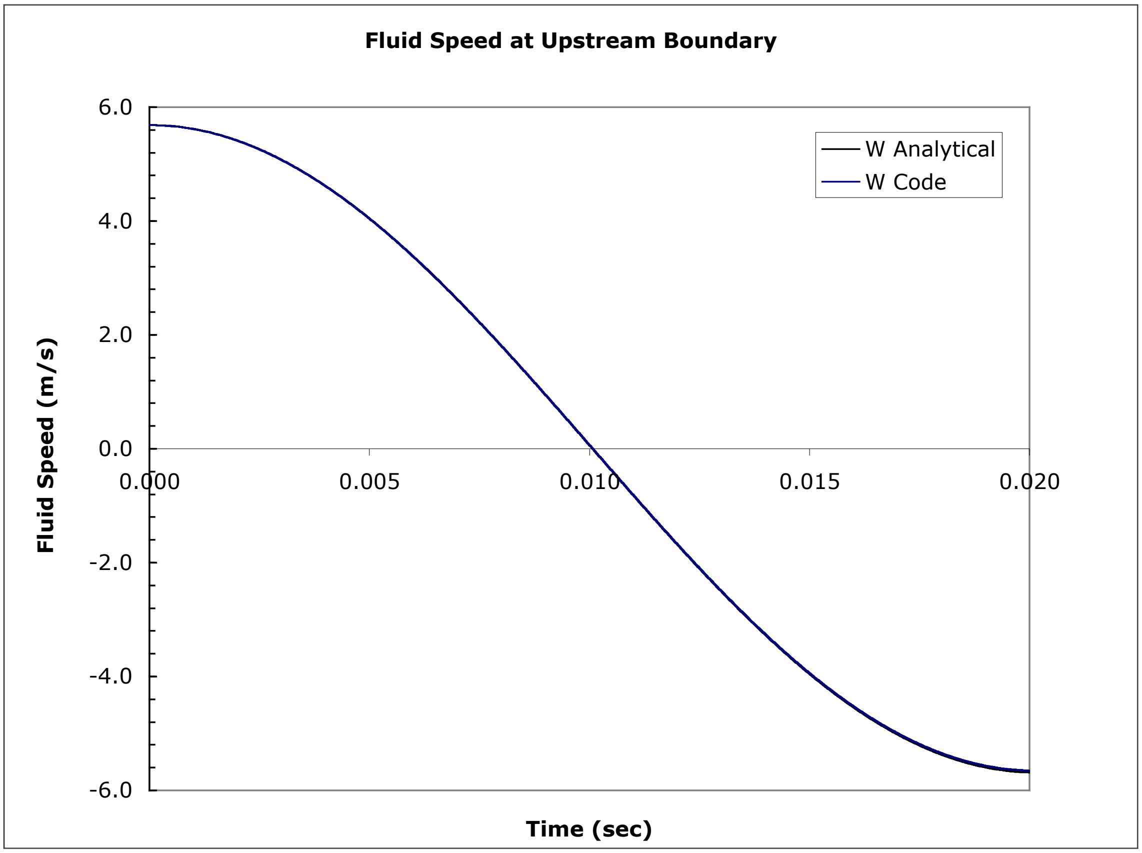

Figure 4. Fluid speed response following instantaneous valve closure.

As shown in Figure 3 and Figure 4, the differences between the analytical solution developed here and the code-calculated results cannot be seen in the Figure. This statement is true for all the comparisons shown in this section.

The response of the volume is given in Figure 5 below.

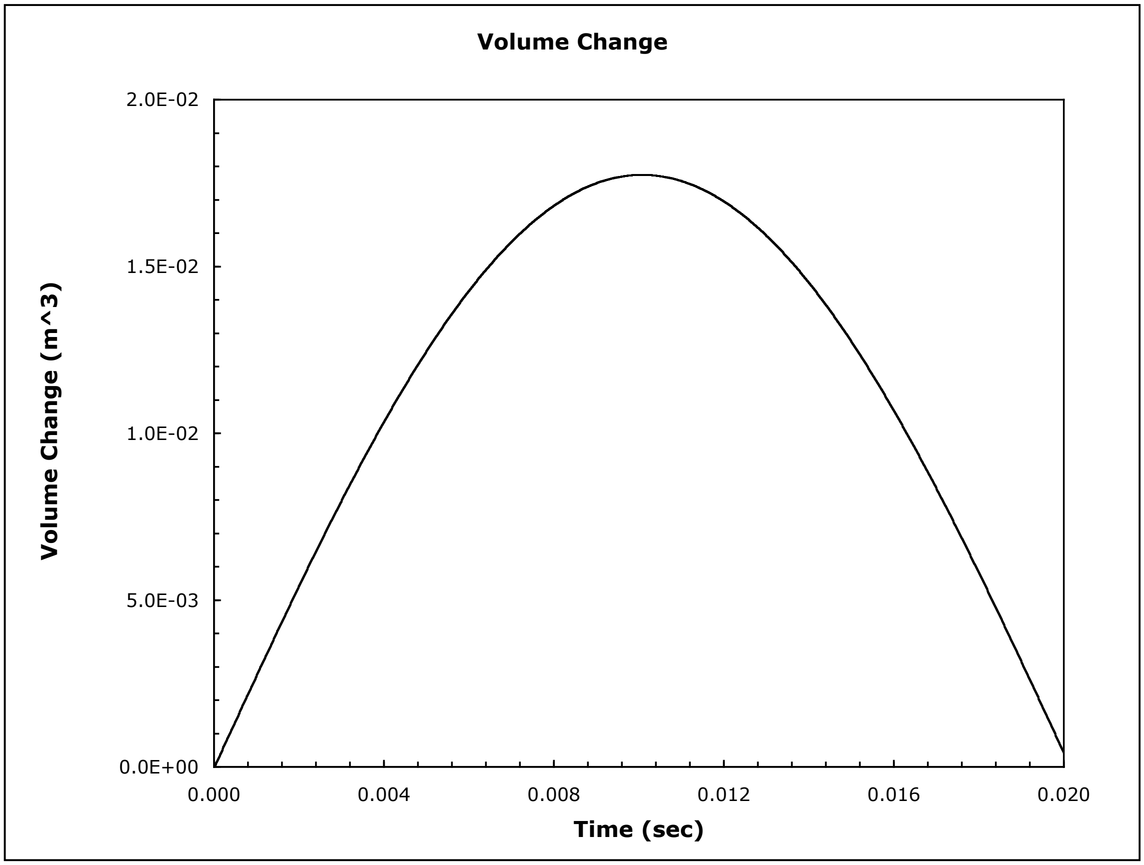

Figure 5. Volume change following valve closure.

Two Nodes Coupled Through a Wall

For the case of two nodes coupled through a deformable wall, the EOS for the pressure for node '*k*' is, Eq. (1.14),

$$P_k = \hat{P}_k(M_k, E_k, M_z, E_z) \tag{1.36}$$

with a similar expression for node '*z*'. Carrying out the procedure used for the single-node case gives

$$\begin{aligned} \frac{d}{dt}P_k = {} & \left(\frac{\partial}{\partial M_k}\hat{P}_k\right)_{\ldots}\frac{d}{dt}M_k + \left(\frac{\partial}{\partial E_k}\hat{P}_k\right)_{\ldots}\frac{d}{dt}E_k \\ & + \left(\frac{\partial}{\partial M_z}\hat{P}_k\right)_{\ldots}\frac{d}{dt}M_z + \left(\frac{\partial}{\partial E_z}\hat{P}_k\right)_{\ldots}\frac{d}{dt}E_z \end{aligned} \tag{1.37}$$

with a similar expression for node 'z'. The wall function for node 'k' for the coupled-node case is

$$V_k = V_{k0} + K_w[\left(P_k(t) - P_k(0)\right) - \left(P_z(t) - P_z(0)\right)] \tag{1.38}$$

The model equations for conservation of mass and energy content must be expanded to include these models for both nodes. To be explicit, and for convenient reference these are

Mass conservation models for the flows are

$$\frac{d}{dt}M_k = W_{ik}(t)$$

and (1.39)

$$\frac{d}{dt}M_z = W_{iz}(t)$$

where M is the mass of fluid in the pipe,

$$M = \rho V \tag{1.40}$$

and $W_i(t)$ is the mass flow rate of fluid into the pipe at the upstream boundary

$$W_i(t) = \rho U A_f \tag{1.41}$$

where ρ is the fluid density, U is the fluid speed at the entrance of the pipe, and A_f is the flow area for the pipe. The initial condition is

$$M(0) = M_0 \tag{1.42}$$

Energy balance model equations are

$$\frac{d}{dt}E_k = h_{ik} W_{ik}(t)$$

and (1.43)

$$\frac{d}{dt}E_z = h_{iz} W_{iz}(t)$$

where h_i is the enthalpy of the fluid entering the pipe at the upstream boundary and the energy content for the fluid is

$$E = Mu = \rho V u \tag{1.44}$$

where u is the specific internal energy for the fluid. The initial condition is

$$E(0) = E_0 \tag{1.45}$$

The pressure-volume work term has been omitted from the energy equation model. Calculations following the solution can be used to show that the term is small and neglecting it is justified.

Neglecting all terms in a momentum balance model except for the pressure gradient, momentum equation models for the fluid flow into the pipes at the upstream end are

$$\frac{d}{dt}W_{ik} = \frac{A_{fk}}{L}\left(P_{upk} - P_k\right)$$

and (1.46)

$$\frac{d}{dt}W_{iz} = \frac{A_{fz}}{L}\left(P_{upz} - P_z\right)$$

where P_{up} is the constant upstream pressure and L is a representative distance for the pressure gradient; one-half the pipe length. The initial condition is

$$W_i(0) = W_i^0 \tag{1.47}$$

Note that wall friction and gravity have not been included. Gravity is easily included, but wall friction is not. For verification purposes, the wall friction can be minimized by use of a few different techniques available in most models and codes; make the pipe diameter be really big, for example.

Putting the appropriate mass and energy equations into the derivative of the pressure, Eq. (1.37) will give two equations for the pressure responses for each node. Those equations can be directly integrated if the mass flow rate into each nodes is a given function of time analogous to Eq. (1.23) way above in these notes. And likewise all the other equations can also be directly integrated. The results are straightforward and will not be summarized here.

Note that several different calculations can be obtained by specialization of the two-node case. Including rigid-wall results by assigning K_w to be zero, and one-node flexible-wall results by making one of the nodes sufficiently large that the fluid pressure remains constant in that node. A null case is obtained by setting geometry, ICs and BCs to be the same for each node. For all the cases considered in these notes, another test can be obtained by changing the finite difference gird in the problem specifications so that a change in the sign of the mass flow is obtained and all else is the same.

The solution for the case of an initial flow is specified and sudden stopping of that flow in either or both of the nodes requires numerical solution of the coupled equations for $P_k(t)$ and $P_z(t)$.

Several numerical-benchmark-grade verification problems can be devised and the ODEs solved by off-the-shelf solver software used to present the expected results. I plan to pursue this aspect if the proposed paper is accepted for publication.

Conclusions

These problems offer several verification opportunities as Method of Exact Solutions (MES) problems for transient, compressible flows both with and without fluid-structure inretactions.