

Chaos and ODEs Part 1b: The Equation Systems

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The General Form of the ODE Systems

The general form of the systems of ODEs is illustrated by the following example of an initial value problem (IVP) with three equations for three dependent variables.

$$\begin{aligned}\frac{dX}{dt} &= \hat{X}(X, Y, Z) \\ \frac{dY}{dt} &= \hat{Y}(X, Y, Z) \\ \text{and} \\ \frac{dZ}{dt} &= \hat{Z}(X, Y, Z)\end{aligned}\tag{0.1}$$

with

$$\begin{aligned}X(t = 0) &= X_0 \\ Y(t = 0) &= Y_0 \\ \text{and} \\ Z(t = 0) &= Z_0\end{aligned}\tag{0.2}$$

given as initial conditions (ICs).

For almost all systems considered in these notes the right-hand sides (RHS) of Eqs. (0.1) are not functions of the independent variable, t . These types of equations are denoted *autonomous* equations. The RHS functions can be linear or non-linear and all of the cases of interest in these notes the RHS will contain some non-linear functions of the dependent variables. Additionally, we may occasionally consider systems made up of more than three equations. Any textbook will provide all detailed information about such equations systems. The objective here is to provide summaries of the information needed for the present purposes. Ascher and Petzold [1998] is a good starting place although a large number of other texts are equally useful.

The equation systems considered in these notes relative to convergence of numerical solution methods are summarized in the following paragraphs.

The Saltzman System

The Saltzman system [1962] is of interest because of historical reasons. The famous Lorenz system of 1963, which has been at the center of dynamical system studies for over four decades now, was derived from the Saltzman system. Saltzman used the first seven letters of the alphabet as symbols for the dependent variables and the equations are

$$\begin{aligned}
\frac{dA}{dt} &= 23.521BC - 1.500D - 148.046A \\
\frac{dB}{dt} &= -22.030AC - 1.589E - 186.429B \\
\frac{dC}{dt} &= 1.561AB - 0.185F - 400.276C \\
\frac{dD}{dt} &= -16.284CE - 16.284BF - 13.958AG - 1460.631\lambda A - 14.805D \\
\frac{dE}{dt} &= 16.284CD - 16.284AF - 18.610BG - 1947.508\lambda B - 18.643E \\
\frac{dF}{dt} &= 16.284AE + 16.284BD - 486.877\lambda C - 40.028F \\
\frac{dG}{dt} &= 27.916AD + 37.220BE - 39.479G
\end{aligned} \tag{0.3}$$

where

$$\lambda = \frac{Ra}{Ra_c}$$

is the ratio of the Rayleigh number to the critical value of the Rayleigh number. The Prandtl number, which always appears explicitly in the Lorenz system, is buried in the numerical values given in the above equations. I have not yet gone over Saltzman's development and attempted to extract the Prandtl number. The numerical value of the Prandtl number in the above equations is $Pr=10.0$; the value that is almost always used with the Lorenz system of 1963. Saltzman noted that this value is about twice that for liquid water; it is also about ten times that of air.

Saltzman discusses the limitations of the system relative to physical reality and suggests that calculations using larger values of the parameter λ are seriously affected by severe truncation of the series used to obtain the equations. Saltzman limited his calculations to $\lambda \leq 10.0$. Tritton has also remarked on the limitations associated with the system that Lorenz developed from Saltzman's system relative to the range of the Rayleigh-number ratio.

I have calculated points on Figure 1 of Saltzman's paper. And I might get around to calculating Figure 3 by the time I get to the results postings.

The Lorenz 1963 System

The original Lorenz system of 1963 was obtained from the Saltzman system above by additional eliminations of terms in the series. The system has been the subject of an almost uncountable number of analyses and publications over the past 44 years. I have found the discussions and analyses by Tritton [1988] and Doering and Gibbon [1995] especially useful. It is generally agreed that the Lorenz 1963 system does not correspond to any known fluid flows. As mentioned above Saltzman noted that the Prandtl number used in the equation system is about twice that for water and about ten times that for

air. The additional truncation of the series used by Lorenz probably implies that calculations should be limited to an even smaller range of the Rayleigh-number ratio.

Whatever the status of the system relative to physical reality, it is always considered to be the system that introduced analysis of chaotic response of complex dynamical systems.

The equations for the original Lorenz system of 1963 are

$$\begin{aligned}\frac{dX}{dt} &= -PrX + PrY \\ \frac{dY}{dt} &= -Y + RaX - XZ \\ \text{and} \\ \frac{dZ}{dt} &= -bZ + XY\end{aligned}\tag{0.4}$$

There have been an almost uncountable number of investigations of the original Lorenz system published over the past almost 45 years now. The majority of these investigations take the following values for the parameters in the system; $Pr = 10.0$, and $b = 8 / 3$. The parameter Ra , the ratio of the Rayleigh number to the critical value of the Rayleigh number, is usually the parameter that is varied so as to investigate different regimes of behavior for the system. A widely used value is $Ra = 28.0$. The value used in this report, and the initial conditions, will be given as the calculated results are presented. Lorenz used initial conditions $(X, Y, Z) = (0.0, 1.0, 0.0)$. The step size for all the calculations was reported to be 0.01 time units.

There have been several applications of the Adomian decomposition method (ADM) [1988] to solving the Lorenz system including Hashim et al. [2006], Guellal et al. [1997], and Biazar et al. [2006].

Young [1966] seems to have been among the first to note potential difficulties relative to numerical solutions of the Lorenz 1963 system. Young noted that the numerical method used by Lorenz, denoted a double-approximation procedure, was second order in the truncation approximation. He proceeded to use an explicit fourth-order Runge-Kutta method to show that the results presented by Lorenz could be made to be more in agreement with the Runge-Kutta results by reducing the step size to 0.0005 for the double-approximation procedure. While Young used several step sizes he did not perform a convergence study, and neither had Lorenz. Young carried out the integration to 20 time units; a value that has added significance at this point in time about 40 years later.

The Lorenz 1984 System

The Lorenz system of [1984] and [1989] and recently the subject of two publications in [2006a] and [2006b], is given by

$$\begin{aligned}\frac{dX}{dt} &= -aX - Y^2 - Z^2 + aF \\ \frac{dY}{dt} &= -Y + XY - bXZ + G \\ \text{and} \\ \frac{dZ}{dt} &= -Z + bXY + XZ\end{aligned}\tag{0.5}$$

In Eqs.(0.5) the quantities F and G can be either functions of the independent variable or treated as constants. We will look at the cases for which they are taken to be constants. Other investigations have looked at cases for which they are functions of time. The usual values for the parameters in Eqs.(0.5) are $a = 0.25$, $b = 4.0$, $G = 1.0$, and $F = 8.0$ to simulate 'Winter' conditions. The case of 'Summer' conditions was not investigated for this report. The specific values used for the parameters, and the initial conditions, for each case of calculated results are given along with the results.

This system has been studied also by Lorenz in 1989 and 2006 as noted above, and using a step size of 0.025 time units. Pielke and Zeng [1994] have given results of very long-term integrations of the system also using 0.025 as the step size. The objective of these latter calculations was to determine if a short-term variation (seasonal variations) can lead to significant long-term variability. Pielke and Zeng integrated the equation system for a time span of about 1100 years: much longer than the investigations by Lorenz 1989.

In general the quantities F and G in these equations can be functions of t , but those cases will not be considered in these notes. The 1984 Lorenz system was developed to explore the effects of heating variability in the atmospheric system over the seasons with the aim to investigate that the climate is or is not intransitive. The general outlook being that intransitivity is unlikely. The seasonal variations of the heating was obtained through the F and G terms in (0.5). As in Lorenz 1989 we have first investigated the calculated behavior for constant values of F and G and we have additionally limited the investigation to the Winter season with $F = 8$ and $G = 1$. Lorenz 1989 found that under these conditions that the calculated numbers for system (0.5) exhibit chaotic response.

The Vadasz System

The original Lorenz system, and some much like the original Lorenz system, continue to appear in the literature. Vadasz and colleagues have developed a Lorenz-like system for fluid flows in porous media [2000a, 2000b, 2001]. The equations are

$$\begin{aligned}\hat{X} &= -\alpha X + \alpha Y \\ \hat{Y} &= -Y + RX - (R-1)XZ \\ \text{and} \\ \hat{Z} &= 4\gamma(XY - Z)\end{aligned}\tag{0.6}$$

Again, the numerical values of the parameters and the ICs will be stated when

numerical results are presented. The system is mathematically equivalent to the original Lorenz 1963 system. For the porous media case $\gamma = 1/2$ and for the non-porous case $\gamma = 2/3$. With γ specified Ra and Pr are determined by

$$R = \frac{4Ra}{27\pi^4}$$

and

$$\alpha = Pr$$
(0.7)

for the non-porous case.

Vadasz and Olek have used the Adomian decomposition method [1988] as well as the usual Runge-Kutta method to solve the equations.

The Rossler System

The Rossler system has been extensively studied in the book by Stuart and Humphries [1996, pp. 530-], Corless et al. [1991], and many other publications. The Rossler system is given by

$$\hat{X} = -Y + Z$$
$$\hat{Y} = aY + X$$

and

$$\hat{Z} = -cZ + XZ + b$$
(0.8)

The values of the parameters in the Rossler system used by Corless et al. Were taken to be $a = 0.20$, $b = 0.40$, and $c = 5.7$. The specific values used for the parameters, and the initial conditions, for each case of calculated results are given along with the results.

Dynamics of Chaotic Spikes

A system reported by Terman [1991] has also been investigated for this report. The equations are

$$\hat{X} = Z - 0.5(X + 0.5) - 2.0Y(X + 0.7) - m_{\infty}(X - 1.0)$$

$$\hat{Y} = 1.15(w_{\infty} - Y)\tau$$

and

$$\hat{Z} = \varepsilon(k - X)$$

where

$$m_{\infty} = \frac{1}{2} \left[1 + \tanh \left(\frac{X + 0.01}{0.15} \right) \right] \quad (0.9)$$

$$w_{\infty} = \frac{1}{2} \left[1 + \tanh \left(\frac{X - 0.1}{0.145} \right) \right]$$

$$\tau = \cosh \left(\frac{X - 0.10}{0.29} \right)$$

For some investigations of chaotic response, the equation for \hat{Z} is replaced by

$$\hat{Z} = -\varepsilon[0.22 + X] \left[\left(\frac{1 + \delta}{2} \right) + \left(\frac{\delta - 1}{2} \right) \tanh \left(\frac{X + 0.23}{0.005} \right) \right] \quad (0.10)$$

Note that the sign between the terms in the second bracket is changed from that in the paper. Personal communications have established that the published sign is not correct.

Values of the parameters k and ε and the initial conditions are given when the calculated results are discussed later below in this report. I have calculated Figures 12, 13 and 15 in the paper.

The Chen System

The Chen dynamical system, Chen and Ueta [1999], and investigated by Noorani et al. [2007]. The equations are

$$\hat{X} = a(Y - X)$$

$$\hat{Y} = (c - a)X - XZ + cY \quad (0.11)$$

and

$$\hat{Z} = XY - bZ$$

where a , b and c are positive parameters. With parameters $a = 35$ and $c = 28$, Eqs. (0.11) exhibit non-chaotic behavior when $b = 12$ and chaotic behavior when $b = 3$. The initial conditions will be specified when the calculated results are discussed below in the report.

The Volterra-Lotka System

A system of four equations, generally known as the Lotka-Volterra models of competition (predator-prey) has also been investigated. The system studied by Vano et al. [2006] and Olek [1994] among others is written in compact form as

$$\frac{dy_i}{dt} = r_i y_i \left(1.0 - \sum a_{ij} y_j \right) \quad (0.12)$$

where

$$r_i = \begin{bmatrix} 1 \\ 0.72 \\ 1.53 \\ 1.27 \end{bmatrix} \quad \text{and} \quad (0.13)$$

$$a_{ij} = \begin{bmatrix} 1 & 1.09 & 1.52 & 0 \\ 0 & 1 & 0.44 & 1.36 \\ 2.33 & 0 & 1 & 0.47 \\ 1.21 & 0.51 & 0.35 & 1 \end{bmatrix}$$

The initial conditions will be specified as the calculational results are presented.

Well, that's about it for equation systems. Of course we have not investigated all systems that exhibit chaotic response, but these given above are a good characteristic cross-section of all others, Next we will look at the numerical solution methods used to integrate the equation systems.