

# Fluid Nodes Coupled by a Common Link

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## Introduction

I have another simple node-link arrangement that has an analytical solution. The model application is a slight extension to that given in the previous notes here:  
<http://edaniel.files.wordpress.com/2011/02/onewallcalcs.pdf> .

A picture of the model is shown in the nearby Figure 1. As shown there, two nodes, labeled 'z' and 'k' are connected by a single link.

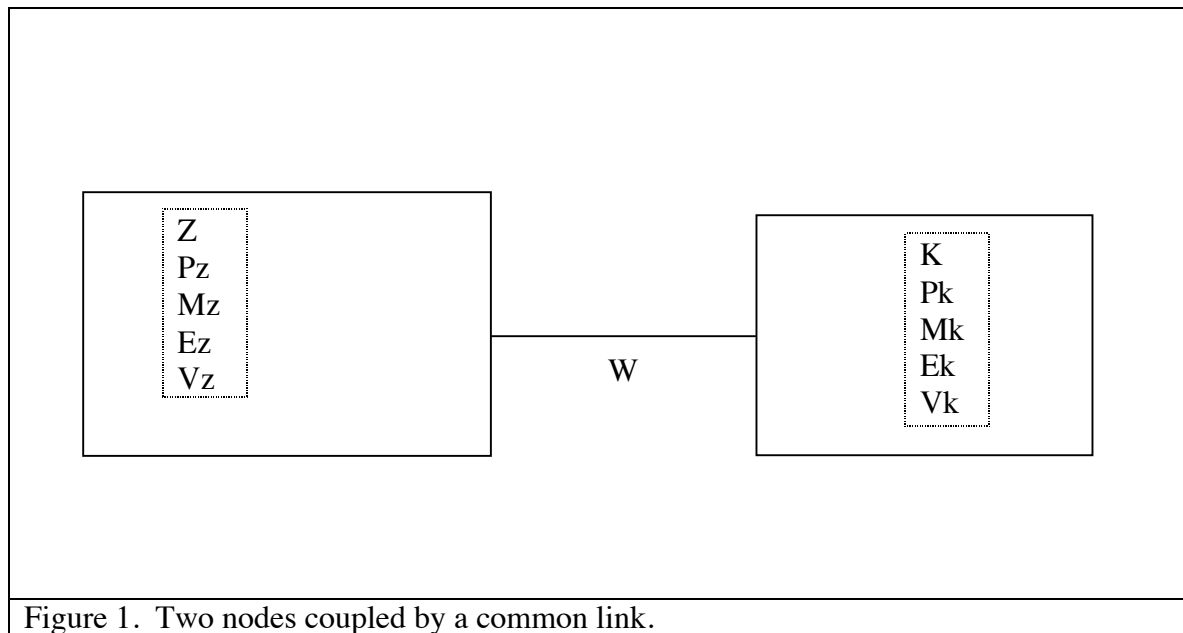


Figure 1. Two nodes coupled by a common link.

The analytical solution is facilitated by the presence of a single, common link.

## The Model Equations

The model equations are as follows.

$$\frac{d}{dt}M_t = W \quad (1.1)$$

$$\frac{d}{dt}M_z = -W$$

for mass conservation

$$\frac{d}{dt}E_k = hW - P \frac{d}{dt}V_k \quad (1.2)$$

$$\frac{d}{dt}E_z = -hW - P \frac{d}{dt}V_z$$

for energy balance. Note that the enthalpy is not associated with either node, but instead is a reference value for both nodes. The initial enthalpy for each node is taken to be the same value, and it is assumed that changes in the enthalpy can be neglected.

The momentum-balance model is

$$\frac{d}{dt}W = \frac{A_f}{L}(P_z - P_k) \quad (1.3)$$

The equivalent inertia for the flow-channel geometry,  $A_f/L$ , is determined by the methods for combining the inertia for node-link combinations that are in series and having different geometry. If all the geometry is the same, then the equivalent inertia is straightforward.

The equation of state returns the pressure given the mass and energy content of each node

$$P_k = \hat{P}_k(M_k, E_k) \quad (1.4)$$

$$P_z = \hat{P}_z(M_z, E_z)$$

The equations of state, Eqs. (1.4) give,

$$\frac{d}{dt}P_k = \left[ \left( \frac{\partial}{\partial M_k} \hat{P}_k \right)_{E_k} + h \left( \frac{\partial}{\partial E_k} \hat{P}_k \right)_{M_k} \right] W = \frac{C_{sfk}^2}{V_k} W \quad (1.5)$$

and

$$\frac{d}{dt} P_z = - \left[ \left( \frac{\partial}{\partial M_z} \hat{P}_z \right)_{E_z} + h \left( \frac{\partial}{\partial E_z} \hat{P}_z \right)_{M_z} \right] W = - \frac{C_{szk}^2}{V_z} W \quad (1.6)$$

with initial conditions, ICs,

$$\left( \frac{d}{dt} P_k \right)^0 = \frac{C_{sfk}^2}{V_{k0}} W^0 \quad (1.7)$$

and

$$\left( \frac{d}{dt} P_z \right)^0 = - \frac{C_{sfz}^2}{V_{z0}} W^0 \quad (1.8)$$

As in the previous notes, the volume of the node is given by the wall function

$$V = V_0 + K_w (P(t) - P(0)) \quad (1.9)$$

where  $K_w$  accounts for the geometry of the node volume and its stress state including the nature of the lateral constraints. A wall function is required for each node.

Note that Eqs. (1.5) and (1.6) have been written in terms of the sound speed for the fluid. As given in the previous notes here: <http://edaniel.files.wordpress.com/2011/02/onewallcalcs.pdf> we know that for the case of deformable / flexible walls the effective sound speed for the wall-fluid combination can be directly substituted into these equations.

The solution for the pressure response is put into Eq. (1.9) to get the volume response, into Eq. (1.3) to get the mass flow rate and those results are put into Eqs. (1.1) to get the mass content, and Eqs. (1.2) to get the energy content response. The last terms on the right-hand side of these latter equations can be either included or omitted; the terms have not been included for these notes.

With the solutions for these equations, other thermodynamic-state properties of interest for the fluid are obtained as follows

$$\rho = \frac{M}{V}$$

for the fluid density

$$u = \frac{E}{M} \tag{1.10}$$

for the specific internal energy, and

$$h = u + P / \rho$$

for the enthalpy.

### The Analytical Solutions

The analytical solutions are obtained as follows; the methodology is exactly the same as presented in previous notes. Taking the time derivative of Eqs. (1.5) and (1.6) gives

$$\frac{d^2}{dt^2} P_k = \frac{C_{sfk}^2}{V_k} \frac{d}{dt} W = \frac{C_{sfk}^2}{V_k} \frac{A_f}{L} (P_z - P_k) \tag{1.11}$$

and

$$\frac{d^2}{dt^2} P_z = -\frac{C_{sfz}^2}{V_z} \frac{d}{dt} W = -\frac{C_{sfz}^2}{V_z} \frac{A_f}{L} (P_z - P_k) \tag{1.12}$$

From Eqs. (1.11) and (1.12) we get

$$\frac{d^2}{dt^2} (P_z + P_k) = \left( \frac{A_f}{L} \right) \left[ \frac{C_{sfk}^2}{V_k} - \frac{C_{sfz}^2}{V_z} \right] (P_z - P_k) \tag{1.13}$$

and

$$\frac{d^2}{dt^2} (P_z - P_k) = -\left( \frac{A_f}{L} \right) \left[ \frac{C_{sfk}^2}{V_k} + \frac{C_{sfz}^2}{V_z} \right] (P_z - P_k) \tag{1.14}$$

Equations (1.13) and (1.14), and the other equations depending on the pressure solution, can be solved by two methods; analytically as they stand and by Laplace transform. I'll use both to provide a check.

Note that if the pressure for one of the nodes is constant, this should reduce to previous results given in the notes here;  
<http://edaniel.files.wordpress.com/2011/02/onewallcalcs.pdf> I have not yet checked on this hypothesis.

Let

$$\begin{aligned}\tau_k &= \frac{C_{sfk}^2}{V_k} \left( \frac{A_f}{L} \right) \\ \tau_z &= \frac{C_{sfz}^2}{V_z} \left( \frac{A_f}{L} \right)\end{aligned}\tag{1.15}$$

and

$$\tau = \sqrt{\tau_k + \tau_z}$$

The solution for Eq. (1.14) for the pressure difference between the nodes is

$$(P_z - P_k) = B_1 \cos \tau t + B_2 \sin \tau t \tag{1.16}$$

with initial conditions

$$(P_z - P_k)_{t=0} = (P_z^0 - P_k^0) = (\Delta P_{zk})^0 \tag{1.17}$$

and, from Eqs. (1.7) and (1.8) for the case of no initial flow,

$$\frac{d}{dt}(P_z - P_k)_{t=0} = 0 \tag{1.18}$$

Note that an initial flow can be easily accommodated in the analytical solution, but that flow will very likely be inconsistent with the initial pressure difference between the nodes.

The ICs give

$$B_1 = (\Delta P_{zk})^0 \tag{1.19}$$

and

$$B_2 = 0 \tag{1.20}$$

and the solution for the pressure difference is

$$(P_z - P_k) = (\Delta P_{zk})^0 \cos \tau t \tag{1.21}$$

where  $\tau = \sqrt{\tau_k + \tau_z}$

Putting the solution Eq. (1.21) into Eq. (1.13) and carrying out the integration gives

$$(P_z + P_k) = B_3 + B_4 t - (\Delta P_{zk})^0 \frac{(\tau_k - \tau_z)}{(\tau_k + \tau_z)} \cos \tau t \quad (1.22)$$

and applying the initial conditions gives

$$\begin{aligned} B_3 &= (P_z + P_k)^0 + (\Delta P_{zk})^0 \\ \text{and} \\ B_4 &= 0 \end{aligned} \quad (1.23)$$

The complete solution is

$$(P_z + P_k) = (P_z + P_k)^0 + (\Delta P_{zk})^0 \hat{\tau} [1 - \cos \tau t] \quad (1.24)$$

where

$$\hat{\tau} = \frac{(\tau_k - \tau_z)}{(\tau_k + \tau_z)}$$

The response for  $P_z$  is obtained by adding the solutions (1.21) and (1.24), and  $P_k$  by subtracting the solutions. These operations give

$$\begin{aligned} P_k &= \frac{1}{\tau_k + \tau_z} \left( \frac{A_f}{L} \right) \left[ \left( \frac{C_{sf}^2}{V} \right)_z P_k^0 + \left( \frac{C_{sf}^2}{V} \right)_k P_z^0 \right] \\ &+ \frac{1}{\tau_k + \tau_z} \left( \frac{A_f}{L} \right) \left( \frac{C_{sf}^2}{V} \right)_k [P_k^0 - P_z^0] \cos \tau t \\ &+ \frac{1}{\tau} \left( \frac{C_{sf}^2}{V} \right)_k W^0 \sin \tau t \end{aligned} \quad (1.25)$$

and

$$\begin{aligned}
P_z = & \frac{1}{\tau_k + \tau_z} \left( \frac{A_f}{L} \right) \left[ \left( \frac{C_{sf}^2}{V} \right)_z P_k^0 + \left( \frac{C_{sf}^2}{V} \right)_k P_z^0 \right] \\
& - \frac{1}{\tau_k + \tau_z} \left( \frac{A_f}{L} \right) \left( \frac{C_{sf}^2}{V} \right)_z [P_k^0 - P_z^0] \cos \tau t \\
& - \frac{1}{\tau} \left( \frac{C_{sf}^2}{V} \right)_z W^0 \sin \tau t
\end{aligned} \tag{1.26}$$

The pressure response is substituted into Eq. (1.9) to get the volume response, and the corresponding equation for node 'z'. I'll check that Eqs. (1.25) and (1.26) reduce to the previous results under suitable conditions.

Putting the pressure response into Eq. (1.3) and carrying out the integration gives the mass flow rate at the link between the nodes

$$W = -\frac{1}{\tau} \left( \frac{A_f}{L} \right) (P_k - P_z) \sin \tau t + W^0 \cos \tau t \tag{1.27}$$

Substituting the mass flow rate into the first of Eqs. (1.1) and integrating gives the mass content for node 'k' which can be written

$$\Delta M_k(t) = \frac{1}{\tau_k + \tau_z} \left( \frac{A_f}{L} \right) (P_k - P_z) \cos \tau t + \frac{1}{\tau} W^0 \sin \tau t - (\Delta M_k(0)) \tag{1.28}$$

with a similar expression for node 'z'. In Eq. (1.28)  $\Delta M_k(t) = M_k(t) - M_k(0)$  is the change in the mass content from the initial time to time 't'.

Neglecting the pressure-volume work term contribution, the change in energy content for node 'k', from the first of Eqs. (1.2) is

$$\Delta E_k(t) = \frac{h}{\tau_k + \tau_z} \left( \frac{A_f}{L} \right) (P_k - P_z) \cos \tau t + \frac{h}{\tau} W^0 \sin \tau t - (\Delta E_k(0)) \tag{1.29}$$

with a similar expression for node 'z'. I have the solution for the case of including the pressure-volume work term, but have not yet reduced it to a nice form. The Laplace transform method was used to verify these solutions.

The solutions given above are valid for the case of a deformable / flexible wall by simply

substituting  $\left( \frac{C_{eff}^2}{V} \right)_k$  for  $\left( \frac{C_{sf}^2}{V} \right)_k$ . Where  $\left( \frac{C_{eff}^2}{V} \right)_k$  is given in the Table 3 of the notes here:

<http://edaniel.files.wordpress.com/2011/02/onewallcalcs.pdf>

### A Few Plots

The solutions can all be investigated by use of a spreadsheet in which the key variables are assigned names so as to have a flexible framework in which to carry out what-if investigations. A couple of cases for numerical evaluation of the solutions are given in the following.

The geometry for each node is the same as given in Table 1 in the notes at the URL just above. The properties of the node walls are the same as given in Table 2, and the initial thermodynamic state and transport properties are the same as given in Table 3 of those notes. A pressure difference is established by setting the pressure for one node different from the value in the Table 3. For the results presented here, the pressure for node ‘z’ is set at  $1.925 \times 10^7$ , so the initial flow is from node ‘z’ into node ‘k’.

As in the original calculations, the node is taken to be a constant-area round pipe 10.0 meters long with a flow area of 1.0 square meter. The dimensions and other properties of the pipe are summarized in nearby Table 1. I don’t have any idea that the values for the diameter and thickness when considering the fluid pressure and temperature used in the calculations make any sense for a real physical pipe.

Dimension	Value
Length (m)	10.00
Flow Area (m <sup>2</sup> )	1.00
Diameter (m)	1.1284
Wall Thickness (m)	0.00635
Table 1. Dimensions of the pipe and pipe wall.	

Pure radial deformation of the wall is assumed, which corresponds to rigid lateral constraints. Models and equations for various other lateral constraints can be found in the literature. For the pure radial deformation case, the wall function

$$V = V_0 + K_w (P(t) - P(0)) \quad (1.30)$$

has

$$K_w = \frac{\pi}{4} D^2 L \left( \frac{1 - \mu^2}{E} \right) \left( \frac{D}{\bar{t}} \right) \quad (1.31)$$

where  $\mu$  is Poisson’s ratio,  $E$  is Young’s modulus, and  $\bar{t}$  is the pipe-wall thickness.

The properties of the wall material are summarized in the nearby Table 2. The numerical values given in Table 2 below correspond to stainless steel, roughly.

<b>Property</b>	<b>Value</b>
Density	7600.0 to 8200.0
Thermal Expansion Coefficient (m/m K)	$1.7 \times 10^{-5}$
Bulk Modulus ( $\text{N/m}^2$ )	$1.66 \times 10^{11}$
Shear Modulus ( $\text{N/m}^2$ )	$7.7 \times 10^{10}$
Tensile Modulus ( $\text{N/m}^2$ )	$1.16 \times 10^{11}$
Young's Modulus ( $\text{N/m}^2$ )	$2.07 \times 10^{11}$
Poisson's Ratio	0.30
$K_{wall}$ ( $\text{m}^5/\text{N}$ )	$7.8118 \times 10^{-9}$
Table 2. Material properties for stainless steel.	

The fluid properties from the NRC/NBS water-property code are summarized in the nearby Table 3.

<b>Property</b>	<b>Value</b>
P (MPa)	0.20
T (K)	350.0
$\rho$ ( $\text{kg/m}^3$ )	973.797
$C_p$ (J/kg K)	4191.25
$C_v$ (J/kg K)	3887.84
$\beta$ (1/K)	$6.2226 \times 10^{-4}$
$\kappa$ ( $\text{m}^2/\text{N}$ )	$4.5869 \times 10^{-10}$
u (J/kg)	$3.2163 \times 10^5$
h (J/kg)	$3.21841 \times 10^5$
$\left(\frac{\partial P}{\partial \rho}\right)_T$ (J/kg)	$2.23878 \times 10^6$

Property	Value
$\left(\frac{\partial P}{\partial T}\right)_\rho$ ( N/m <sup>2</sup> K )	1.3566 x 10 <sup>6</sup>
Table 3. Water properties.	

For the first demonstration calculation, the initial pressure for node ‘z’ is set to a value, 2.25x10<sup>6</sup> Pa, which larger than the value for node ‘k’ and the initial flow is out of the former node into the latter. The first calculation is for rigid-wall nodes and the sound speed for the fluid alone enters the time constants.

The pressure response for the nodes is shown in nearby Figure 2.

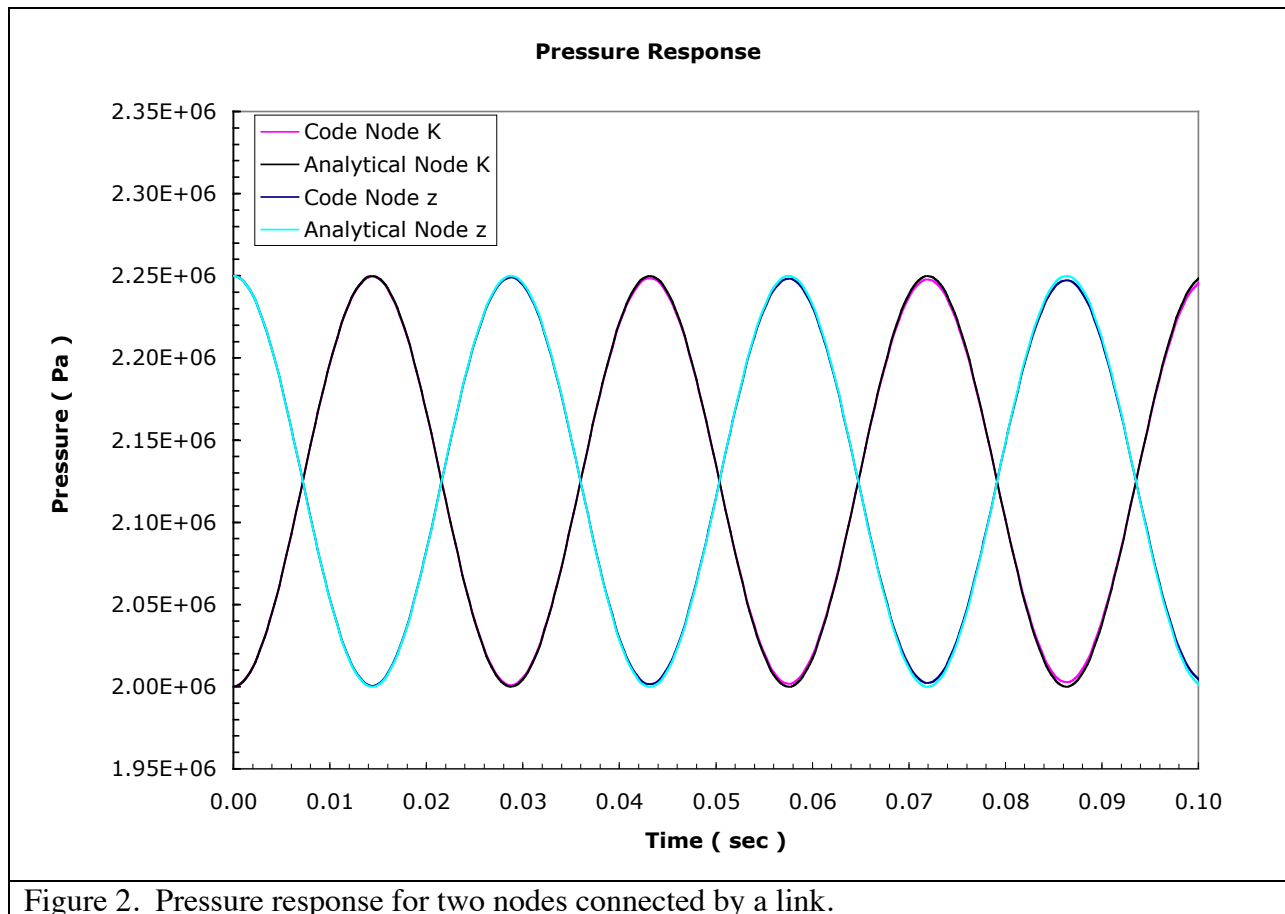


Figure 2. Pressure response for two nodes connected by a link.

The results calculated by a code that is designed for applications much more general than this simple verification problem are also shown in Figure 2. A close examination of the output

printed by the code and compared with the analytical solution shows that the amplitude of the oscillations is decreasing as time increases. This trend is due to numerical artifacts introduced by the numerical solution method used in the code. The analytical solution method can also be used to investigate this property of numerical solution methods.

The fluid speed at the link between the nodes is shown in Figure 3.

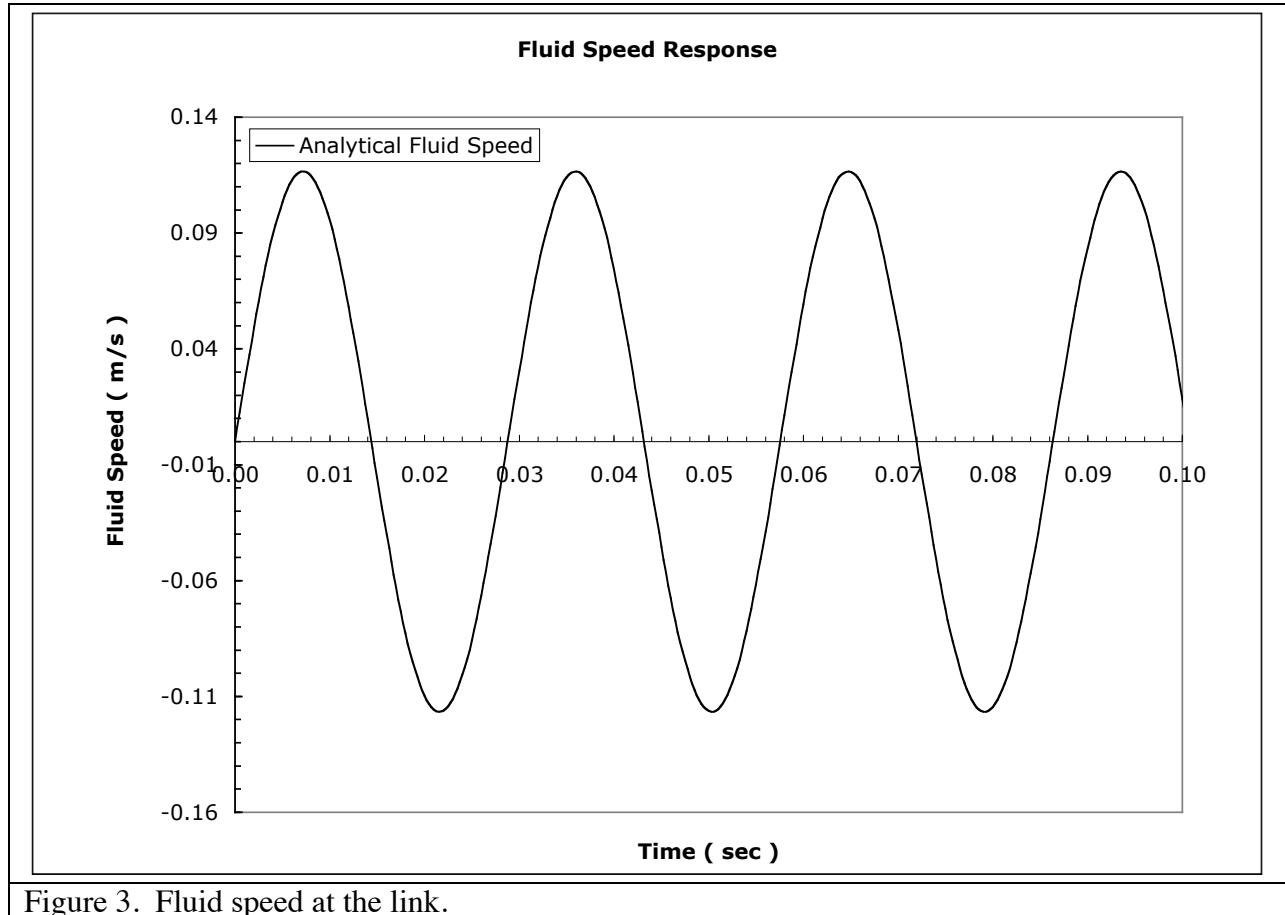


Figure 3. Fluid speed at the link.

For a second calculation, the node walls are taken to be deformable / flexible with different values of  $K_w$  for Eq. (1.30), and thus different values for the effective sound speed. For node 'k', the effective sound speed is taken to be 500.0 m/s and for node 'z' 2000.0 m/s. The pressure response is shown in Figure 4. Node 'z' is much more stiff than node 'k' and so the pressure response is much more pronounced. I have calculated the  $K_w$  values that correspond to these effective sound speeds and so the change in the volume is not presented.

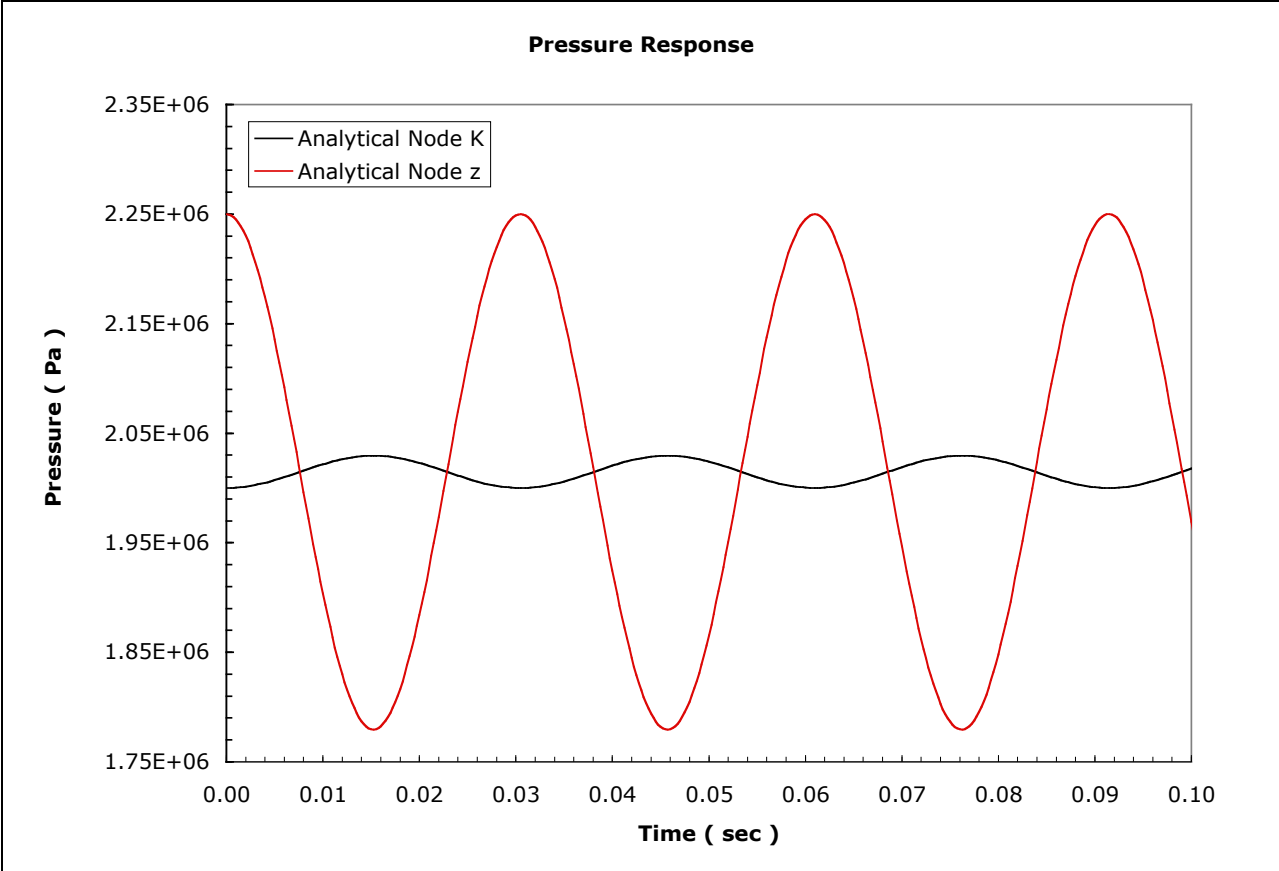


Figure 4. Pressure response for two nodes with flexible walls.